# STRONG FUNCTIONAL NÖRLUND SUMMABILITY

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In this paper relations between 'strong functional' and 'functional' Nörlund summability methods have been established. Some inclusion theorems have also been established which show that if one 'functional' Nörlund method includes the other, then the same is true of the associated 'strong functional' Nörlund methods.

## 1. INTRODUCTION

A definition of strong functional Nörlund summability has been given by Kumar (1974) and some multiplication theorems concerning strong functional Nörlund summability of the Cauchy product of two integrals have been established.

In the present paper, we establish relations between strong functional and functional Nörlund summability methods. We also prove certain inclusion theorems which show that if one functional Nörlund method includes the other then the same is true of the associated strong functional Nörlund methods. Analogous results for strong Nörlund summability of series (and sequences) are contained in Borwein and Cass (1968).

## 2. PRELIMINARIES

Let S be the class of (complex valued) functions $s(t)$ of a real variable $t$ defined for all positive $t$ and bounded and measurable in every finite interval $(0, T)$, $T > 0$.

Let $\sigma(t)$ be the integral transform of $s(t) \in S$ defined by

$$\sigma(t) = \int_0^\infty a(t, u)\, s(u)\, du. \qquad \ldots(2\cdot1)$$

The transformation (2·1) with kernel $a(t, u)$ is said to be regular over the set $S$, if $s(t) \to s$ implies $\sigma(t) \to s$ as $t \to \infty$; and is called null-preserving if $s(t) \to 0$ implies $\sigma(t) \to 0$ as $t \to \infty$. The necessary and sufficient conditions for the regularity of (2·1) over the set $S$ are (Hardy 1949, 50, p. 61):

$$\text{(i)} \quad \int_0^\infty |a(t, u)|\, du = O(1), \qquad \ldots(2\cdot2)$$

(ii) $$\int_0^\infty a(t, u)\, du \to 1 \text{ as } t \to \infty, \qquad \ldots(2{\cdot}3)$$

(iii) $$\int_E a(t, u)\, du \to 0 \text{ as } t \to \infty \qquad \ldots(2{\cdot}4)$$

for every bounded and measurable set $E$ of $u$-axis. But if $a(t, u)$ is non-negative. then (iii) is equivalent to

(iv) $$\int_0^c a(t, u)\, du \to 0 \text{ as } t \to \infty \qquad \ldots(2{\cdot}5)$$

for every finite $c > 0$.

The conditions (i) and (iii) are necessary and sufficient for the transformation (2·1) to be null-preserving.

(a) *Functional Nörlund Summability* $(N, p, \beta)$ (see Choudhary 1970)

Let $p(t)$ be Lebesgue integrable in any (relevant) finite interval (*). We write

$$p_1(t) = \beta + \int_0^t p(u)\, du$$

and assume that, for sufficiently large $t$,

$$p_1(t) \neq 0.$$

The $(N, p, \beta)$ transform of $s(t) \in S$ is defined by

$$\rho(t) = \frac{1}{p_1(t)} \{\beta\, s(t) + \int_0^t p(t-u)\, s(u)\, du\}. \qquad \ldots(2{\cdot}6)$$

If $\rho(t) \to s$ as $t \to \infty$, we say $s(t)$ is summable by the Nörlund method $(N, p, \beta)$ to $s$, and denote this by

$$s(t) \to s\,(N, p, \beta).$$

When $\beta = 0$, this definition reduces to the customary definition of functional Nörlund summability $(N, p)$ (see Knopp and Vanderburg 1955).

We recall that, with the usual notation for convolutions, the integral in (2·6) can be written $(p * s)_t$. We shall make use of the result that the convolutions are associative and commutative, *i. e.*,

$$a * b = b * a \text{ and } a * (b * c) = (a * b) * c.$$

(*) This assumption about $p(t)$ (and $q(t)$ as well) is made throughout.

(b) *Strong Functional Nörlund Summability* $[N, p, \beta]_\lambda$, $\lambda > 0$

Let $\mathfrak{F}$ be the class of functions $f(t)$ which is an indefinite integral of some Lebesgue integrable function, say $a(t)$, *i.e.*,

$$f(t) = f(0) + \int_0^t a(u)\, du.$$

Let $p(t)$ satisfy the following: For given $T > 0$ there exists $\mu = \mu(T) > 0$ such that

$$| p(t) | \geqslant \mu \quad (0 \leqslant t \leqslant T). \qquad \ldots(2 \cdot 7)$$

We describe $f(t) \in \mathfrak{F}$ as strongly summable $(N, p, \beta)$ with index $\lambda > 0$ to the value $s$, and write $f(t) \rightarrow s\, [N, p, \beta]_\lambda$, if

$$\int_0^t | p(u) |^{1-\lambda} \, | Y(u) - [s - f(0)]\, p(u) |^{\lambda}\, du = o\, [\, | p_1(t) | \,]$$

where

$$Y(t) = \beta\, a(t) + (p * a)_t. \qquad \ldots(2 \cdot 8)$$

In the special case in which $\beta = 0$ and $f(0) = 0$, this definition can easily be seen to be equivalent to that given by Kumar (1974).

*Remarks*: As remarked by Kumar (1974), $p(t)$ should be assumed to satisfy (2·7) only in the case when $\lambda > 1$. Further, whenever we shall be concerned with strong Nörlund summability with index $> 1$, it will be automatically assumed that the generating function satisfies (2·7) and will not be stated explicitly.

For two given summability methods $D$ and $E$, we say that the method $E$ includes the method $D$ if every function summable $D$ is also summable $E$ to the same sum and write $D \subseteq E$. And we say that $D$ and $E$ are equivalent if each includes the other and write $D = E$.

## 3. Relations between Strong Functional and Functional Nörlund Summability

***Theorem* 1** — If

$$| p_1(t) | \rightarrow \infty \text{ as } t \rightarrow \infty, \qquad \ldots(3 \cdot 1)$$

then

$$[N, p, \beta]_1 \subseteq (N, p, \beta).$$

Proof: Suppose that $f(t) \rightarrow s[N, p, \beta]_1$. Thus we are given that

$$\frac{1}{| p_1(t) |} \int_0^t | Y(u) - [s - f(0)]\, p(u) |\, du = o\,(1). \qquad \ldots(3 \cdot 2)$$

Now

$$\frac{1}{p_1(t)} \int_0^t [\, Y(u) - [s - f(0)]\, p(u)\,]\, du = X(t) - s + \frac{\beta\,[s - f(0)]}{p_1(t)}$$

where

$$X(t) = \frac{1}{p_1(t)}\,[\beta\, f(t) + (p * f)_t]. \qquad \ldots(3\cdot3)$$

Thus

$$|\, X(t) - s \,| \leqslant \frac{1}{|\, p_1(t)\,|} \int_0^t |\, Y(u) - [s - f(0)]\, p(u)\,|\, du + \frac{|\,\beta\,[s - f(0)]\,|}{|\, p_1(t)\,|}$$

$$= o(1)$$

by (3·2) and (3·1). Hence $f(t) \rightarrow s\,(N, p, \beta)$ and the required inclusion follows.

*Theorem* 2—If (3·1) holds and

$$p_1^*(t) = O\,(\,|\, p_1(t)\,|\,) \qquad \ldots(3\cdot4)$$

where

$$p_1^*(t) = \int_0^t |\, p(u)\,|\, du$$

then

$[N, p, \beta]_\lambda \subsetneq (N, p, \beta)$ for $\lambda > 1$.

PROOF: Firstly we show that, for $\lambda > \mu > 0$,

$$[N, p, \beta]_\lambda \subsetneq [N, p, \beta]_\mu. \qquad \ldots(3\cdot5)$$

Let $f(t) \rightarrow s\,[N, p, \beta]_\lambda$. Thus we are given that

$$\frac{1}{|\, p_1(t)\,|} \int_0^t |\, p(u)\,|^{1-\lambda}\, |\, Y(u) - [s - f(0)]\, p(u)\,|^{\lambda}\, du = o\,(1). \qquad \ldots(3\cdot6)$$

By Hölder's inequality, we obtain

$$\frac{1}{|\, p_1(t)\,|} \int_0^t |\, p(u)\,|^{1-\mu}\, |\, Y(u) - [s - f(0)]\, p(u)\,|^{\mu}\, du$$

$$< \left[ \frac{1}{|\, p_1(t)\,|} \int_0^t |\, p(u)\,|^{1-\lambda}\, |\, Y(u) - [s - f(0)]\, p(u)\,|^{\lambda}\, du \right]^{\mu/\lambda} \times \left[ \frac{p^*_1(t)}{|\, p_1(t)\,|} \right]^{(\lambda - \mu)/\lambda}$$

$$= o\,(1)$$

by (3·4) and (3·6). Hence $f(t) \rightarrow$ s $[N, p, \beta]_\mu$ and the inclusion in (3·5) is thus established.

The result now follows from (3·5) and Theorem 1.

*Theorem* 3—If (3·1) and (3·4) hold, and $\lambda \geqslant 1$, then for $f(t) \to s\ [N, p, \beta]_\lambda$ it is necessary and sufficient that $f(t) \to s\ (N, p, \beta)$ and that

$$\frac{1}{|p_1(t)|} \int_0^t |p(u)|^{1-\lambda} \ |Y(u) + f(0)\, p(u) - p(u)\, X(u)|^{\lambda}\, du = o(1) \qquad \ldots(3\cdot7)$$

where $Y(t)$ and $X(t)$ are respectively given by (2·8) and (3·3).

PROOF: *Necessity*—Suppose that $f(t) \to s\ [N, p, \beta]_\lambda$. Then, by Theorems 1 and 2, $f(t) \to s\ (N, p, \beta)$. Thus, in this case, we have

$$\frac{1}{|p_1(t)|} \int_0^t |p(u)|^{1-\lambda} \ |Y(u) - [s - f(0)]\, p(u)|^{\lambda}\, du = o(1) \qquad \ldots(3\cdot8)$$

and

$$X(t) \to s \quad \text{as} \quad t \to \infty. \qquad \ldots(3\cdot9)$$

It follows from (3·1) and the result

$$p_1^*(t) \geqslant |p_1(t) - \beta|$$

(which at once follows from the definition of $p_1(t)$) that

$$p_1^*(t) \to \infty \quad \text{as} \quad t \to \infty. \qquad \ldots(3\cdot10)$$

By virtue of (3·9), we have

$$\frac{1}{p_1^*(t)} \int_0^t |p(u)| \ |X(u) - s|^{\lambda}\, du = o(1) \qquad \ldots(3\cdot11)$$

because the transformation on the left of (3·11) with kernel

$$a(t, u) = \begin{cases} \dfrac{|p(u)|}{p_1^*(t)} & \text{for} \quad 0 < u \leqslant t \\ 0 & \text{for} \quad u > t \end{cases}$$

can easily be seen to be regular whenever (3·10) holds. Further, by (3·4), (3·11) is equivalent to

$$\frac{1}{|p_1(t)|} \int_0^t |p(u)| \ |X(u) - s|^{\lambda} = o(1). \qquad \ldots(3\cdot12)$$

By Minkowski's inequality, we obtain

$$\left[\frac{1}{|p_1(t)|}\int_0^t |p(u)|^{1-\lambda}\ |Y(u)+f(0)\,p(u)-p(u)\,X(u)|^{\lambda}\,du\right]^{1/\lambda}$$

$$\leqslant \left[\frac{1}{|p_1(t)|}\int_0^t |p(u)|^{1-\lambda}\ |Y(u)-[s-f(0)]\,p(u)|^{\lambda}\,du\right]^{1/\lambda}$$

$$+\left[\frac{1}{|p_1(t)|}\int_0^t |p(u)|\ |X(u)-s|^{\lambda}\right]^{1/\lambda}. \qquad \ldots(3{\cdot}13)$$

Using (3·8) and (3·12) in (3·13), we obtain (3·7).

*Sufficiency*—Suppose that (3·7) and (3·9) hold. Then, in this case also, (3·12) holds. Hence, by Minkowski's inequality,

$$\left[\frac{1}{|p_1(t)|}\int_0^t |p(u)|^{1-\lambda}\ |Y(u)-[s-f(0)]\,p(u)|^{\lambda}\,d_u\right]^{1\lambda}$$

$$\leqslant \left[\frac{1}{|p_1(t)|}\int_0^t |p(u)|^{1-\lambda}\ |Y(u)+f(0)\,p(u)-X(u)\,p(u)|^{\lambda}\,du\right]^{1/\lambda}$$

$$+\left[\frac{1}{|p_1(t)|}\int_0^t |p(u)|\ |X(u)-s|^{\lambda}\,du\right]^{1\lambda}$$

$$= o(1)$$

by (3·7) and (3·12). Thus (3·8) holds and hence $f(t)\to s\,[N, p, \beta]_\lambda$. This completes the proof of the theorem.

## 4. Inclusion Theorems

*Theorem* 4—Let $p(t)$ and $q(t)$ satisfy these hypotheses:

$$\left.\begin{array}{lll} p(t)=h(t)\,t^{\xi}, & h(t)\in C^{n+1\,(*)}, & h(0)=1 \\ q(t)=l(t)\,t^{\eta}, & l(t)\in C^{n+1}, & l(0)=1 \end{array}\right\} \qquad \ldots(4{\cdot}1)$$

where $\eta \geqslant \xi$ and where $n$ is the least integer $\geqslant \eta$, $\xi > -1$. Suppose that $p(t) > 0$, $q(t) > 0$ for all $t$ and $p_1(t)\to\infty$ as $t\to\infty$ where

$$p_1(t)=\int_0^t p(u)\,du.$$

(*) The symbol $C^n$ stands for the class of functions possessing continuous derivatives of the $n$th order if $t \geqslant 0$.

Suppose also that $(N, p)$ and $(N, q)$ are regular. Then

$(N, p) \subseteq (N, q)$ implies $[N, p]_1 \subseteq [N, q]_1$.

PROOF: By Theorem 9 of Knopp and Vanderburg (1955), if $\eta \geqslant \xi$ there exists a continuous function $k(t)$ such that

$$q(t) = p(t) + (p * k)_t \quad \text{if } \eta = \xi \qquad \text{...(4·2)}$$

or

$$q(t) = (p * \{C_{\xi,\eta} t^{\eta-\xi-1} + k\})_t \quad \text{if } \eta > \xi \qquad \text{...(4·3)}$$

where $C_{\xi,\eta} = B^{-1}(\xi + 1, \eta - \xi)$.

If we write

$$K(t) = \begin{cases} k(t) & \text{if } \eta = \xi \\ C_{\xi,\eta}\, t^{\eta-\xi-1} + k(t) & \text{if } \eta > \xi \end{cases} \qquad \text{...(4·4)}$$

then (4·2) and (4·3) may be replaced by the single relation

$$q(t) = e\, p(t) + (p * K)_t \qquad (\text{for } t > 0) \qquad \text{...(4·5)}$$

where $e=1$ or $0$ according as $\eta = \xi$ or $\eta > \xi$ respectively. Further, by Theorem 11 of Knopp and Vanderburg (1955) a necessary condition for $(N, p) \subseteq (N, q)$ is

$$\int_0^t p_1(u) \,|\, K(t-u) \,|\, du = O(q_1(t)). \qquad \text{...(4·6)}$$

Under the given hypotheses, from (4·6), we have (cf. the proof of Theorem 12 of Knopp and Varderburg 1955)

$$p_1(t) = O(q_1(t)) \qquad \text{...(4·7)}$$

and

$$\int_0^c |\, K(t-u) \,|\, du = o(q_1(t)) \qquad \text{...(4·8)}$$

for every finite $c > 0$.

Write

$$M(t) = (p * a)_t - (s - f(0))\, p(t), \qquad \text{...(4·9)}$$

$$N(t) = (q * a)_t - (s - f(0))\, q(t), \qquad \text{...(4·10)}$$

$$\delta(t) = \frac{1}{p_1(t)} \int_0^t |\, M(u) \,|\, du. \qquad \text{...(4·11)}$$

Now, suppose that $f(t) \to s\, [N, p]_1$. Thus, we are given that

$$\delta(t) = o(1). \qquad \text{...(4·12)}$$

Also, by (4·5) and (4·9),

$$N(t) = e\, M(t) + (K * M)_t, \qquad \ldots(4\cdot13)$$

and hence, by (4·11),

$$\frac{1}{q_1(t)} \int_0^t |N(u)|\, du \leqslant e\, \frac{p_1(t)}{q_1(t)}\, \delta(t) + \frac{1}{q_1(t)} \int_0^t |K(v)|\, dv \int_0^{t-v} |M(w)|\, dw$$

$$= e\, \frac{p_1(t)}{q_1(t)}\, \delta(t) + \frac{1}{q_1(t)} \int_0^t p_1(v)\, |K(t-v)|\, \delta(v)\, dv. \qquad \ldots(4\cdot14)$$

The first term on the right side of (4·14) is $o(1)$ by (4·7) and (4·12). And the second term is a null-preserving transform for, in this case, (4·6) and (4·8) correspond to (2·2) and (2·5). Hence, since $\delta(t) = o(1)$, the second term is also $o(1)$. Thus

$$\frac{1}{q_1(t)} \int_0^t |N(u)|\, du = o(1).$$

Hence $f(t) \to s\ [N, q]_1$ and the required inclusion follows.

*Corollary*—If, in addition to the hypotheses of Theorem 4, $q_1(t) \to \infty$ as $t \to \infty$, then

$(N, p) = (N, q)$ implies $[N, p]_1 = [N, q]_1$.

*Note*: The corollary is valid only when $\eta = \xi$, since $(N, p) = (N, q)$ implies $\eta = \xi$ (see Theorem 13 of Knopp and Vanderburg 1955).

*Theorem* 5—Suppose that the hypotheses of Theorem 4 are satisfied. Suppose also that

$$\int_0^t p(u)\, |K(t-u)|\, du = O(q(t)) \qquad \ldots(4\cdot15)$$

where $K(t)$ is given by (4·4). Then, whenever $\eta > \xi$,

$(N, p) \subseteq (N, q)$ implies $[N, p]_\lambda \subseteq [N, q]_\lambda$ for $\lambda > 1$.

PROOF: Let $M(t)$ and $N(t)$ be defined by (4·9) and (4·10). Suppose that $f(t) \to s[N, p]_\lambda$. Thus we are given that

$$a(t) = \frac{1}{p_1(t)} \int_0^t \{p(u)\}^{1-\lambda}\, |M(u)|^{\lambda}\, du = o(1). \qquad \ldots(4\cdot16)$$

If we write

$$Z(t) = \frac{(p * a)_t}{p(t)} - (s - f(0)),$$

then $p(t) Z(t) = M(t)$ and so (4·13) reduces to

$$N(t) = (K * pZ)_t$$

since, in this case, $e = 0$. Using Hölder's inequality, we obtain

$$|N(t)|^\lambda \leqslant \{\int_0^t p(u) \,|K(t-u)|\, du\}^{\lambda-1} \{\int_0^t |K(t-u)|\, p(u) |Z(u)|^\lambda du\}$$

$$\leqslant H \{q(t)\}^{\lambda-1} \int_0^t |K(t-u)|\, p(u)\, |Z(u)|^\lambda \, du$$

by (4·15), where $H$ is a positive constant. Thus

$$\{q(t)\}^{1-\lambda} \,|N(t)|^\lambda \leqslant H \int_0^t |K(t-u)| \,\{p(u)\}^{1-\lambda}\, |M(u)|^\lambda du$$

and hence

$$\frac{1}{q_1(t)} \int_0^t \{q(u)\}^{1-\lambda} \,|N(u)|^\lambda \, du$$

$$\leqslant \frac{H}{q_1(t)} \int_0^t |K(v)| \int_0^{t-v} \{p(w)\}^{1-\lambda} \,|M(w)|^\lambda \, dw\, dv$$

$$= \frac{H}{q_1(t)} \int_0^t |K(t-v)|\, p_1(v)\, \alpha(v)\, dv \qquad \ldots(4·17)$$

$$= o(1)$$

by (4·16) and the fact that (4·17) is a null-preserving transform (cf. the proof of Theorem 4). Thus $f(t) \to s\ [N, q]^\lambda$ and the required inclusion follows.

*Theorem* 6—In addition to the hypotheses of Theorem 4, suppose that $r(t) > 0$ for all $t > 0$ and that

$$r_1(t) \to \nu < \infty \text{ as } t \to \infty. \qquad \ldots(4·18)$$

Then

$$(N, p) \subseteq (N, q) \text{ implies } [N, p * r]_1 \subseteq [N, q * r]_1.$$

PROOF: Let

$$\bar{M}(t) = (p * r * a)_t - (s - f(0))\,(p * r)_t,$$

$$\bar{N}(t) = (q * r * a)_t - (s - f(0))\,(q * r)_t,$$

$$\gamma(t) = \frac{1}{(1 * p * r)_t} \int_0^1 |\bar{M}(u)|\, du. \qquad \ldots(4·19)$$

Suppose that $f(t) \to s\,[N, p, * r]_1$. Then

$$\gamma(t) = o(1). \qquad \ldots(4{\cdot}20)$$

With the aid of (4·5), we find

$$\overline{N}(t) = e\,\overline{M}(t) + (K * \overline{M})_t$$

and thus, using (4·19), we get

$$\frac{1}{(1 * q * r)_t} \int_0^t |\,\overline{N}(u)\,du$$

$$\leqslant \frac{e\,(1 * p * r)_t}{(1 * q * r)_t}\,\gamma(t) + \frac{1}{(1 * q * r)_t} \int_0^t |\,K(t-u)\,|\,(1 * p * r)_u\,\gamma(u)\,du$$

$$\leqslant \frac{r_1(t)}{r_1(1)} \;\}\; \{ \; \frac{q_1(t)}{q_1(t-1)} \;\}\; [ \; \frac{e p_1(t)}{q_1(t)}\,\gamma(t) +$$

$$+ \frac{1}{q_1(t)} \int_0^t |\,K(t-u)\,|\,p_1(u)\,\gamma(u)\,du] \qquad \ldots(4{\cdot}21)$$

since

$$(1 * p * r)_t = \int_0^t p(t-u)\,r_1(u)\,du \leqslant p_1(t)\,r_1(t)$$

and

$$(1 * q * r)_t \geqslant \int_1^t q(t-u)\,r_1(u)\,du \leqslant q_1(t-1)\,r_1(1).$$

By virtue of (4·20), the expression in the bracket [ ] in (4·21) is $o(1)$ (cf. the proof of Theorem 4). Also

$$\frac{q_1(t)}{q_1(t-1)} \to 1 \quad \text{as } t \to \infty$$

by the regularity of $(N, q)$. Thus, by (4·18), we obtain

$$\frac{1}{(1 * q * r)_t} \int_0^t |\,\overline{N}\,|\,(u)\,du = O(1).$$

This implies $f(t) \to s\,[N, q{*}r]_1$ and the result follows.

The proof of the next theorem can be closely modelled on the proofs of Theorems 5 and 6.

*Theorem* 7—If, in addition to the hypotheses of Theorem 6,

$$\int_0^t (p * r)_u\,|\,K(t-u)\,|\,du = O\,[\,(q * r)_t\,] \qquad \ldots(4{\cdot}22)$$

where $K(t)$ is given by (4·4), then, whenever $\eta > \xi$,

$$(N, p) \subseteq (N, q) \text{ implies } [N, p^*r]_\lambda \subseteq [N, q^*r]_\lambda \text{ for } \lambda > 1.$$

We remark that Theorems 4 and 6 are independent in the sense that they are not deducible from one another. A similar remark applies to Theorems 5 and 7.

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