ON ABSOLUTE ABEL SUMMABILITY FACTORS IN A SEQUENCE

by Z. U. AHMAD *and* MOHD. MOBIN-UR-RAHMAN KHAN, *Department of Mathematics, Aligarh Muslim University, Aligarh*

(Communicated by R. S. Mishra, F.N.A.)

(*Received* 8 *September* 1975)

In this paper, a theorem for absolute Abel summability factors for sequences analogous to that of a result of Tatchell for series, has been established. It is expected that it may be useful in proving a similar result of Tyler for absolute Cesàro summability, when the orders of summability are non-integral.

1. DEFINITIONS AND NOTATIONS

Let $\sum a_n$ be a given infinite series and let $\{s_n\}$ be a sequence of real numbers, not necessarily the sequence of partial sums of $\sum a_n$. Let us write.

$$\alpha_s(x) = \sum_{n=0}^{\infty} a_n x^n = (1-x) \sum s_n x^n. \quad \text{...(1·1)}$$

The series $\sum a_n$, or the sequence $\{s_n\}$, is said to be absolutely Abel summable, or simply summable $| A |$, if the series on the right of (1·1) converges for $0 \leqslant x < 1$, and $\alpha_s(x) \in BV[0, 1)$, (see Whittaker 1930).

2. INTRODUCTION

On the suggestion of Professor Bosanquet, Tyler (1958) established the following theorem on the absolute Cesàro summability factors for sequences.

Theorem A—(Tyler 1958, Theorem 1)—If k, $\rho \geqslant 0$ (κ, ρ integers), then necessary and sufficient conditions for the sequence $\{\varepsilon_n\}$ to be such that $\{s_n \varepsilon_n\}$ is summable $| C, \rho |$ whenever $\{s_n\}$ is summable $| C, k |$, are

(i-*a*) $\quad \varepsilon_n = O(n^{\rho-k})$

(i-*b*) $\quad \sum_{r=0}^{n} \varepsilon_r = O(n)$

(ii) $\quad \triangle^{k-1} \varepsilon_n = O(n^{1-k})$

(iii) $\quad \sum_{n=0}^{\infty} | \triangle C^{\rho}(\varepsilon_n) | < \infty$, *i.e.* $\{\varepsilon_n\}$ is summable $| C, \rho |$.

It is to be noted that this theorem is an analogue of a particular case of a result of Bosanquet (1954) for ordinary Cesàro summability.

As Theorem A has been proved only for integral order of summability, the question arises: What will be its proof when ρ, k are non-integral? The proof seems to be complicated and, perhaps, requires a number of auxiliary results and the use of functional analysis techniques.

In this paper we prove a theorem for absolute Abel summability factors for sequences, as an analogue of the following result of Tatchell for ordinary Abel summability established by him in certain other context, and we think that it may be of some help in tackling our question.

Theorem B (Tatchell 1954, Theorem 1)—Necessary and sufficient conditions for $\sum_0^\infty a_n \varepsilon_n$ to be summable $|A|$ whenever $\sum a_n$ is convergent are that

(i) $\sum_1^\infty |\Delta \varepsilon_n| < \infty$ and

(ii) $\sum_1^\infty n^{-1} |\varepsilon_n| < \infty$.

3. Theorem

We prove the following theorem:

Theorem—Necessary and sufficient conditions for $\{s_n \varepsilon_n\}$ to be summable $|A|$ whenever $\{s_n\}$ is convergent are:

(i) $\sum_{n=1}^\infty n^{-1} |\varepsilon_n| < \infty$

and

(ii) $\{\varepsilon_n\}$ is summable $|A|$.

4. Lemmas

We require the following lemmas for the proof of our theorem:

Lemma 1 (Tatchell 1954, Lemma 2)—If a sequence $\{\varepsilon_n\}$ has the property that the function

$$\sum_0^\infty s_n \varepsilon_n \left\{ \frac{d}{dx} (1 - x) x^n \right\}$$

is defined and has a finite Lebesgue integral on [0, 1) whenever $\{s_n\}$ is a convergent sequence, then there is a number H such that

$$\int_0^1 \left| \sum s_n \varepsilon_n \left\{ \frac{d}{dx} (1 - x) x^n \right\} \right| dx \leqslant H \,\overline{bd}\, |s_n|,$$

for every sequence $\{s_n\}$.

Lemma 2 (Orlicz 1929, Satz 2)*—If a sequence $\{ p_n \}$ of elements in a Banach space B (see Banach 1932) has the property that there is a number H such that

$$\| \sum_{0}^{k} \pm p_n \| \leqslant H,$$

for each k and every set of signs $\pm$, then

$$\sum_{0}^{\infty} | f(p_n) | < \infty,$$

for every linear functional f on B.

5. Proof of the Theorem

Necessity—Since, by definition, Abel transform of the sequence $\{ s_n \varepsilon_n \}$ is given by

$$\alpha_{s\epsilon} (x) \equiv (1 - x) \sum_{n=0}^{\infty} s_n \varepsilon_n x^n$$

and since $\{ s_n \}$ is convergent, taking $s_n = 1$, for each $n = 0, 1, 2,\ldots\ldots$, we see that the condition (ii) is necessary, for $\alpha_{s\epsilon} (x) \in BV$ [0, 1).

Again, since $\{ s_n \}$ is convergent, suppose $\lim_{n \to \infty} s_n = s$, and put $s'_n = s_n - s$. Then

$$\alpha_{s\epsilon}(x) = (1 - x) \sum_{n=0}^{\infty} (s_n - s) \varepsilon_n x^n + (1 - x) \sum_{n=0}^{\infty} \varepsilon_n x^n$$

$$= (1 - x) \sum_{n=0}^{\infty} s'_n \varepsilon_n x^n + (1 - x) \sum_{n=0}^{\infty} \varepsilon_n x^n$$

$$= \alpha^*_{s\epsilon}(x) + \alpha_\epsilon(x) \qquad \ldots(5{\cdot}1)$$

say, where

$$\alpha^*_{s\epsilon} (x) = (1 - x) \sum_{n=0}^{\infty} s'_n \varepsilon_n x^n$$

and

$$\alpha_\epsilon (x) = (1 - x) \sum_{n=0}^{\infty} \varepsilon_n x^n.$$

Now, since

$$\alpha_\epsilon (x) \in BV [0, 1) \qquad \ldots(5{\cdot}2)$$

* The two conditions in the enunciation of the lemma are, in fact, equivalent (see Tatchell 1954, footnote on p. 208).

by condition (ii), $a_{s\epsilon}(x) \in BV\,[0, 1)$ implies that

$$\int_0^1 \left| \sum s'_n \, \varepsilon_n \left\{ \frac{d}{dx} \, (1 - x)\, x^n \right\} \right| dx$$

$$= \int_0^1 \left| \frac{d}{dx} \left\{ \sum s'_n \, \varepsilon_n \, (1 - x)\, x^n \right\} \right| dx$$

$$= \int_0^1 \left| d \sum s'_n \, \varepsilon_n \, (1 - x)\, x^n \right| < \infty \qquad \ldots(5{\cdot}3)$$

for every convergent sequence $\{ s'_n \}$.

Following Tatchell (1954), we see that it follows from Lemma 1 that (5·3) holds only if there is a number H such that

$$\int_0^1 \left| \sum s'_n \, \varepsilon_n \left\{ \frac{d}{dx} (1 - x)\, x^n \right\} \right| dx$$

$$\leqslant H \, \overline{bd} \, | s'_n |,$$

for every sequence $\{ s'_n \}$. In particular (5·3) implies that

$$\int_0^1 \left| \sum_{n=0}^{k} \pm \varepsilon_n \left\{ \frac{d}{dx} (1 - x)\, x^n \right\} \right| dx \leqslant H$$

for each k and every set of signs $\pm$ and so, by Lemma 2,

$$\sum | \varepsilon_n | \left| \int_0^1 \phi\,(x) \left\{ \frac{d}{dx} (1 - x)\, x^n \right\} dx \right| < \infty \qquad \ldots(5.4)$$

for every bounded real function $\phi(x)$.

We do not impose any additional restriction by assuming that (5·4) holds for every bounded complex function $\phi(x)$. We may assume, in particular, that (5·4) holds with $\phi(x) = (1 - x)^i$. Interpreting the integral in the Cauchy-Lebesgue sense and integrating by parts, we obtain, for $n > 0$,

$$\left| \int_0^1 (1 - x)^i \left\{ \frac{d}{dx} \, (1 - x)\, x^n \right\} dx \right|$$

$$= \left| i \int_0^1 (1 - x)^i \, x^n \, dx \right|$$

$$= \left| \frac{i\,\Gamma(1 + i)\ \Gamma(n + 1)}{\Gamma(n + 2 + i)} \right|$$

$$\sim | \Gamma\,(1 + i) | \, n^{-1}$$

as $n \to \infty$, where $\Gamma(1 + i) \neq 0$. Therefore, by (5·4), condition (i) is necessary.

Sufficiency—Condition (i) ensures that $\alpha_{s\epsilon}(x)$ is defined on [0, 1) whenever $\{s_n\}$ is convergent (since (i) implies that $\epsilon_n = O(n)$ and hence $|\alpha_{s\epsilon}(x)| \leqslant \sum_{n=0}^{\infty} |s_n| \, |\epsilon_n| \, x^n$ $\leqslant K(|\epsilon_0| + \sum_{n=1}^{\infty} n x^n) \leqslant K[|\epsilon_0| + \sum_{n=1}^{\infty} (n+1) x^n] \leqslant K$, for $0 \leqslant x < 1$).

Conditions (i) and (ii) ensure that (5·1) holds and $\alpha_{s\epsilon}(x) \in BV\,[0, 1)$, since by condition (i) we have

$$\int_0^1 |d\{\alpha^*_{s\epsilon}(x)\}| = \int_0^1 |d\{\Sigma\, s_n'\, \epsilon_n\, (1-x)\, x^n\}|$$

$$\leqslant \sum |s_n'| \, |\epsilon_n| \int_0^1 |d\{(1-x)\, x^n\}|$$

$$\leqslant \overline{bd}\, |s_n'| \left\{ |\epsilon_0| + \sum_{n=1}^{\infty} n^{-1} |\epsilon_n| \right\}$$

$$< \infty .$$

This completes the proof of the theorem.

REFERENCES

Banach, S. (1932). Thèorie des operations linèaire. Warsaw.

Bosanquet, L. S. (1954). On convergence and summability factors in a sequence. *Mathematika*, **1**, 24-44.

Orlicz, M. W. (1929). " Beiträge zur theorie der orthogonalent-wichlungen II. *Studia Mathematica*, **1**, 241-55.

Tatchell, J. B. (1954). A note on a theorem of Bosanquet. *J. Lond. math. Soc.*, **29**, 207-11.

Tyler, B. (1958). Absolute convergence and summability factors in a sequence. *J. Lond. math. Soc.*, **33**, 341-51.

Whittaker, J. M. (1930). The absolute summability of Fourier series. *Proc. Edin. math. Soc.* (2), **2**, 1-5.