THE n-DIMENSIONAL GENERALIZED STIELTJES TRANSFORMATION OF DISTRIBUTIONS

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The Stieltjes transform arises naturally as an iteration of the Laplace transform. Some generalizations of the conventional Stieltjes transform were given by mathematicians from time to time. In this paper three of these generalizations are mentioned. Two of these with the conventional Stieltjes transform are discussed in the generalized function sense for the case of n dimensions. The first of these generalizations is developed in detail and the second one is discussed rather in a brief way. A number of testing function spaces are introduced along with their duals and some results involving some of these spaces are discussed. Important properties like boundedness, analyticity and certain asymptotic order relations are also given. We conclude with a remark about the third generalization.

1. INTRODUCTION

Widder (1946) made a detailed study of the conventional Stieltjes transform

$$F(s) = \int_0^\infty (s + t)^{-1} f(t)\, dt \qquad \ldots(1)$$

and Benedetto (1967) discussed its analytic representation in the generalized function sense. Pandey (1972) has recently developed the theory of (1) in the generalized function sense. The Stieltjes transformation, as a special case of the convolution transformation, has been studied by Zemanian (1968 section 8·6). The n-dimensional distributional Laplace transformation was developed with an apparently new approach by Zemanian (1966) and the n-dimensional Weierstrass transformation was studied by Queen (1971). Recently Rao (1975) discussed the n-dimensional generalized Laplace transform of certain distributions by employing the testing function space $L_{A,B}$ and its dual $L'_{A,B}$.

A generalization of (1) studied by Arya (1958) is the following:

$$F(s) = \Gamma(2m + 1)\, \Gamma^{-1}(m - k + 3/2)\, s^{-1}$$

$$\times \int_0^\infty F(2m + 1,\ 1;\ m - k + 3/2;\ - t/s)\, f(t)\, dt \qquad \ldots(2)$$

where $F(2m + 1,\ 1;\ m - k + 3/2;\ - t/s)$ is a hypergeometric function and $\mathrm{Re}\,(2m + 1) > 0$. For $k + m = \frac{1}{2}$, (2) reduces to (1).

Another generalization of (1) studied by Golas (1967, 1970) is the following:

$$F(s) = \Gamma(b + c + 1)\,\Gamma(b + 1)\,\Gamma^{-1}(a + b + c + 1)\,s^{-1}$$
$$\times \int_0^\infty (t/s)^b F(b + c + 1, b + 1; a + b + c + 1; -t/s) f(t)\,dt. \quad \ldots(3)$$

where $b \geqslant 0$, $a + b + c + 1 \neq 0, -1, -2, \ldots$. Equation (3) reduces to (1) for $a = b = 0$ and also reduces to (2) for $b = 0$, $a = \frac{1}{2} - m - k$, $c = 2m$. For $k - m = \frac{1}{2}$ and $2m + 1 = \sigma$, (2) reduces to a generalization studied by Widder (1946, p. 328.) namely

$$F(s) = \int_0^\infty (s + t)^{-\sigma} f(t)\,dt\ (\sigma > 0). \quad \ldots(4)$$

Another generalization of (1) with σ being replaced by $\varrho + 1$ in (4) was studied in the distributional sense by Misra (1972).

We define the one-dimensional generalized Stieltjes transformation of a generalized function $f(t)$ directly as the application of $f(t)$ to $g(st) = \Gamma(2m + 1) \times \Gamma^{-1}(m - k + 3/2)s^{-1} F(2m + 1, 1; m - k + 3/2; -t/s)$ in the following way:

$$F(s) = \langle f(t), g(st) \rangle .$$

This requires the construction of a space of testing functions on $0 < t < \infty$ which contains $g(st)$ for various value of s.

2. Notation and Terminology

The notation follows that of Pandey (1972), Rao (1975) and Zamanian (1966). Throughout this paper I denotes the open interval $(0, \infty)$. $\langle f, \phi \rangle$ denotes the number assigned to some ϕ in a testing function space by a member of the dual space. The spaces D, $D(I)$, S, $E(I)$ and their duals have their usual meaning (Pandey 1972, Rao 1975, Zamanian 1966). We assign to all these spaces their customary topologies (Schwartz 1966, pp. 88-90).

In what follows let

$$t = \{ t_1, t_2, \ldots, t_n \} \in R^n$$
$$s = \{ s_1, s_2, \ldots, s_n \} \in C^n$$
$$[\, F(2m + 1, 1; m - k + 3/2; -t/s)\,] = F(2m + 1, 1;$$
$$m - k + 3/2; -t_1/s_1) F(2m + 1, 1; m - k + 3/2; -t_2/s_2) \ldots$$
$$\ldots F(2m + 1, 1; m - k + 3/2; -t_n/s_n).$$

$f(t)$, $[t^{\alpha}]$, $[s^{-1}]$ and D_t^{ν} have the same meaning as in Rao (1975) and Zamanian (1966). $P = \Gamma(2m + 1)\,\Gamma^{-1}(m - k + 3/2)$. Throughout this work it is assumed that $\mathrm{Re}\,(2m + 1) > 0$ and $m - k + 3/2 \neq 0, -1, -2 \ldots$.

3. The Testing Function Space S_α and its Dual

Let t, $\alpha \in R^n$, $s \in C^n$ and t_j be an arbitrary component of t. Let $f(t)$ be a function from R^n into C^1. The conventional n-dimensional generalized Stieltjes transformation maps a suitably restricted function of this sort into another function $F(s)$ where $F(s)$ maps C^n into C^1 through the integral

$$F(s) = \int_{R^n} f(t)\, g(st)\, dt.$$

Setting $K_\alpha(t) = \prod_{\nu=1}^{n} k_{\alpha_\nu}(t_\nu)$

where $k_{\alpha_\nu}(t_\nu) = t_\nu^{\alpha_\nu}$, $0 < t_\nu < \infty$ $(\alpha_\nu < 0)$.

We use the symbol S_α to denote the space of all complex-valued smooth functions $\phi(t)$ from R^n into C^1 such that, for every integer $\nu \geqslant 0$ $(\nu \in R^n)$

$$\gamma_\nu(\phi) = \sup_{0 < t < \infty} | K_\alpha(t)\, D_t^\nu\, \phi(t) | < \infty \qquad \ldots(5)$$

for all $\nu = 0, 1, 2, \ldots$.

We see that γ_0 is a norm and hence the collection of semi-norms $\{\gamma_\nu\}_{\nu=0}^{\infty}$ given by (5) is separating (Zamanian 1968). The topology in S_α is generated by the collection of semi-norms $\{\gamma_\nu\}_{\nu=0}^{\infty}$ (Zamanian 1968). The notion of convergence in S_α and the Cauchy sequence are the same as in Pandey (1972). S_α is a vector space with respect to the field of complex numbers. It is also a sequentially complete countably multinormed space.

S'_α is the dual space of S_α and the following properties of S_α and S'_α are straightforward.

(i) $D(I)$ is a subspace of S_α and the topology of $D(I)$ is stronger than the topology induced on $D(I)$ by S_α.

(ii) S_α is a dense subspace of $E(I)$.

$D(I) \subset S_\alpha \subset E(I)$

(iii) $E'(I) \subset S'_\alpha \subset D'(I)$.

(iv) *Boundedness property of generalized functions in* S'_α — Let $f \in S'_\alpha$. There exists a non-negative integer $r \in R^1$ and a finite positive constant $C_2 \in R^1$ such that, for all $\phi \in S_\alpha$,

$$|\langle f, \phi \rangle| \leqslant C_2 \max_{0 \leq |l| \leq r}\ \sup_{0 < t < \infty} | K_\alpha(t) D_t^l\, \phi(t) | .$$

Lemma 3·1—Let

$g(st) = P(s)^{-1} F(2m + 1, 1; m - k + 3/2; -t/s)$.

Then for a complex s not lying on the real axis

$(\sigma \leqslant 0, \tau = 0$ where $s = \sigma + i\tau)$

$$[g(st)] = P[s^{-1}][F(2m+1, 1; m-k+(3/2); -t/s)]$$

belongs to S_{α} under the conditions (A)

(i) $2m$ is zero or a positive integer

or (ii) $2m \neq 0$ or a positive integer, $k \pm m \neq \frac{1}{2}$, Re $m \geqslant 0$,

or(iii) $k + m = \frac{1}{2}$; alternatively under the conditions (B)

(i) $k - m = \frac{1}{2}$ or (ii) $2m \neq 0$ or a positive integer Re $m < 0$.

PROOF: $$|K_{\alpha}(t)\, D_t^{\nu}[g(st)]| = \Big| P[s^{-1}]\, K_{\alpha}(t) \times \frac{\partial^{\nu_1}}{\partial t_1^{\nu_1}} \left\{ F\left(2m+1, 1; m-k+\frac{3}{2}; -t_1/s_1\right) \right\} \times \frac{\partial^{\nu_2}}{\partial t_2^{\nu_2}} \left\{ F\left(2m+1, 1; m-k+\frac{3}{2}; -t_2/s_2\right) \right\} \cdots \times \frac{\partial^{\nu_n}}{\partial t_n^{\nu_n}} \left\{ F\left(2m+1, 1; m-k+\frac{3}{2}; -t_n/s_n\right) \right\} \Big| .$$

Let us consider

$$\left| P(s_1^{-1})\, K_{\alpha_1}(t_1) \frac{\partial^{\nu_1}}{\partial t_1^{\nu_1}} \left\{ F(2m+1, 1; m-k+(3/2); -t_1/s_1) \right\} \right|$$
$$= | PM(s_1^{-1})\, K\alpha_1(t_1)\, s_1^{-\nu_1} F(2m+1+\nu_1, 1+\nu_1; m-k+(3/2)+\nu_1; -t_1/s_1) | \quad \ldots(6)$$

by using the differential property of hypergeometric function [Erdelyi 1953, p. 102, (20)] where M is a constant involving m, k, ν_1. A typical term of (6) containing s_1 and t_1 namely

$$t_1^{\alpha_1} F(2m+1+\nu_1, 1+\nu_1; m-k+(3/2)+\nu_1; -t_1/s_1)\, s_1^{-1-\nu_1} \quad \ldots(7)$$

is asymptotic [Arya 1958, p. 120], [Erdelyi 1953, p. 63] to

$$s_1^{-1-\nu_1} t_1^{\alpha_1} (t_1/s_1)^{-1-\nu_1} \quad \ldots(8)$$

under the conditions (A). Again (7) is asymptotic (Arya 1958, Erdelyi 1953) to

$$s_1^{-1-\nu_1} t_1^{\alpha_1} (t_1/s_1)^{-(2m+1+\nu_1)} \quad \ldots(9)$$

under the conditions (B). Hence Lemma 3·1 is proved since expressions in (8) and (9) tend to 0 under the given conditions.

Lemma 3·2—Let σ, $\alpha \in R^n$ where σ is the real part of s. Let the conditions (A) hold; alternatively let the conditions (B) hold. Then, if

(i) $\theta \in S$, then $\theta\Big[g(\sigma t)\Big] \in S_{\alpha}$;

(ii) $\left\{ \theta_j \right\}_{j=1}^{\infty}$ converges to 0 in S, then

$\left\{ \left[g(\sigma t) \right] \theta_j \right\}_{j=1}^{\infty}$ also converges in S_a to 0.

PROOF: We prove (ii) first. We have $K_a(t)\, D_t^{\nu} \left\{ \theta_j(t) \left[g(\sigma t) \right] \right\}$

$$= K_a(t) \left\{ \sum_{r=0}^{\nu} \binom{\nu}{r} M_r \left[\sigma^{-1} \right] \left[\sigma^{-r} \right] \times P \left[F(2m+1+r, 1+r; m-k+3/2+r; -t/\sigma \right] \times \theta_j^{(\nu-r)}(t) \right\} \qquad \ldots(10)$$

where M_r is a constant involving m, k and r. As in the proof of Lemma 3·1, considering a typical term on the right-hand side of (10) and using the asymptotic formulas (Arya 1958, Erdelyi 1953) under conditions (A) and (B) we can prove (ii) in view of the uniform convergence of $\{ \theta_j \}_{j=1}^{\infty}$ for all t in $0 < t < \infty$. The following results can be easily proved:

(i) $[g(st)] f \in S'$ if and only if $f(t)/K_a(t) \in S'$.

(ii) If $f \in S'_a$, then $f(t) [g(st)] \in S'$.

4. Some Further Results

Theorem 4·1—Let $f \in S'_a$ for a fixed s lying in the compact subset of the complex plane not containing the origin. Let $F(s)$ be defined in the generalized function sense by the equation

$$F(s) = \langle f(t), [g(st)] \rangle. \qquad \ldots(11)$$

Then $F(s)$ is an analytic function of s and

$$F^{(\nu)}(s) = \langle f(t), \frac{\partial^{\nu}}{\partial s^{\nu}} \left[g(st) \right] \rangle. \qquad \ldots(12)$$

PROOF: We have already seen that $[g(st)] \in S_a$ for all t in $0 < t < \infty$ and for a fixed Re s. Hence (11) has a meaning. We can show that (12) also has a meaning by using Cauchy's integral formula and using Hartog's theorem (Bochner and Martin 1948). The proof is similar to that followed in Theorem 1 of Pandey (1969) and hence is omitted.

Corollary—If $F^{(\nu)}(x)$ be defined for positive real x as a special case of (12) and $m - k + (3/2) + l \neq 0, -1, -2, \ldots$ ($0 \leq l \leq r$); $r = 0, 1, 2, 3, \ldots$) then the following asymptotic order relations are true

$$| F^{(\nu)}(x) | = o([x^{-\nu}]) \; (x \to \infty, \nu = 1, 2, \ldots) \qquad \ldots(13)$$

$$| F^{(\nu)}(x) | = o([x^{\nu-1}]) \; (x \to 0+, \nu = 1, 2, \ldots) \quad \ldots(14)$$

under conditions (A) specified previously and $0 < x < 1$.

$$| F^{(\nu)}(x) | = o([x^{2m-\nu}]) \; (x \to 0+, \nu = 1, 2, \ldots) \quad \ldots(15)$$

under conditions (B) specified previously and $0 < x < 1$.

PROOF: We have

$$| F^{(\nu)}(x) | = \left< f(t) \; \frac{\partial^{\nu}}{\partial x^{\nu}} \; P \left[x^{-1} \right] [F(2m+1, 1; m-k+(3/2); -t/x)] \right>$$

$$= \left< f(t), P(-1)^{\nu} \Gamma(\nu+1) [x^{-\nu-1}] \right.$$

$$\left. \times [F(2m+1, \nu+1; m-k+3/2; -t/x)] \right>$$

by a result given by Arya (1958, p. 122) and Erdelyi [1953, p. 102, (21)]. Using the boundedness property of generalized functions in S'_{a}. We have

$$\left| \left[x^{\nu} \right] F^{(\nu)}(x) \right| \leqslant C_2 \max_{0 \leqslant |l| \leqslant r} \sup_{0 < t < \infty} \left\{ \left[t^{\alpha} \right] \right.$$

$$\left. \times \left[x^{-l-1} \right] M \left[F(2m+1+l, l+\nu+1; m-k+(3/2)+l; -t/x) \right] \right\}$$

(here it is understood that l traverses all integers in R^n for which $0 \leqslant |l| \leqslant r$) where M is a constant involving l, m, k and ν.

We note that $F(2m+1+l, \nu+l+1; m-k+(3/2)+l; -t/x)$ tends to 0 as $x \to \infty$. Proof of (13) is now obvious. From asymptotic formula (Arya 1958; Erdelyi 1953, p. 63) under conditions (A) we see that $| [x^{-\nu+1}] F^{(\nu)}(x) | \leqslant C_2 \max_{0 \leqslant |l| \leqslant r} \sup_{0 < t < \infty} \{ [x^{-1}] \times M [t^{\alpha-\nu-l-1}] \}$.

The right-hand side expression is obviously bounded as $x \to 0+ \; (0 < x < 1)$ and $a - \nu - l - 1 < 0$ for all t in $0 < t < \infty$. Hence (14) is proved. Using the asymptotic formula under conditions (B) we have

$$| [x^{\nu-2m}] F^{(\nu)}(x) | \leqslant C_2 \max \sup [t^{\alpha-2m-1-l}] [x^{l}] M.$$

The right-hand side expression $\to 0$ as $x \to 0+ \; (0 < x < 1)$ for any t in $0 < t < \infty$. (15) is therefore proved.

5. Study of the Transform (1)

In this section we propose to state the results relating to the transform (1) along with brief proofs wherever necessary.

We write (1) as $F(s) = \dfrac{1}{s} \displaystyle\int_0^{\infty} \left(1 + \frac{t}{s} \right)^{-1} f(t) \, dt.$

$$g(st) = \frac{1}{s} \left(1 + \frac{t}{s} \right)^{-1}.$$

We write (1) in the distributional sense as

$F(s) = < f(t), g(st) >$. We will set $\left[\left(1 + \frac{t}{s} \right)^{-1} \right]$

$$= \left(1 + \frac{t_1}{s_1} \right)^{-1} \left(1 + \frac{t_2}{s_2} \right)^{-1} \cdots \left(1 + \frac{t_n}{s_n} \right)^{-1}.$$

With the same definitions of $K_{\boldsymbol{a}}(t)$, seminorms (5), testing functions space $S_{\boldsymbol{a}}$, convergence in $S_{\boldsymbol{a}}$, its relations with other function spaces and boundedness property of generalized functions in $S'_{\boldsymbol{a}}$ as described in Section 3, we have

Lemma 5·1—Let $g(st) = \frac{1}{s} \left(1 + \frac{t}{s} \right)^{-1}$.

Then for a complex s not lying on the real axis ($\sigma \leqslant 0$, $\tau = 0$ where $s = \sigma + i\tau$)

$$\left[g(st) \right] = \left[s^{-1} \right] \left[\left(1 + \frac{t}{s} \right)^{-1} \right]$$

belongs to $S_{\boldsymbol{a}}$.

PROOF: We have $| K_{\boldsymbol{a}}(t)\, D_t^{\nu} [g(st)] | = \left| [t^{\boldsymbol{a}}] [s^{-1}] \frac{\partial^{\nu}}{\partial t^{\nu}} \left[\left(1 + \frac{t}{s} \right)^{-1} \right] \right|$.

Considering that

$$| t_1^{a_1} s_1^{-1-\nu_1} \left(1 + \frac{t_1}{s_1} \right)^{-(\nu_1 + 1)} (-1)^{\nu_1} \nu_1! | < t_1^{a_1 - \nu_1 - 1} \nu_1!$$

we can complete the proof of Lemma 5·1.

Lemma 5·2—Let σ, $\boldsymbol{a} \in R^n$ where σ is the real part of s. Then if

(i) $\theta \in S$ then $\theta [g(\sigma t)] \in S_{\boldsymbol{a}}$;

(ii) $\{ \theta_j \}^{\infty}_{j=1}$ converges to 0 in S, then

$\{g(\sigma t)\, \theta_j\}^{\infty}_{j=1}$ also converges in $S_{\boldsymbol{a}}$ to 0.

Proof: We prove (ii) first. By noting that as in the proof of Lemma 3·2,

$$K_{\boldsymbol{a}_1}(t_1)\, \sigma_1^{-1-r_1} \left(1 + \frac{t_1}{\sigma_1} \right)^{-(r_1 + 1)} \theta_j^{(\nu_1 - r_1)}(t_1)$$

$$< t_1^{\boldsymbol{a}_1}\, t_1^{-r_1-1}\, \theta_j^{(\nu_1 - r_1)}(t_1)$$

which is bounded for all t_1 in $0 < t_1 < \infty$ and for any σ_1 since $\boldsymbol{a}_1 - r_1 - 1 < 0$. In the above inequality r_1 is an integer taking value from 0 to ν_1. Since $\{ \theta_j \}_{j=1}^{\infty}$ converges uniformly to 0 for all t_1 in $0 < t_1 < \infty$, $\{[g(\sigma t)]\, \theta_j\}_{j=1}^{\infty}$ also converges to 0 in $S_{\boldsymbol{a}}$. Proof of Lemma 5·2 is complete.

The two results (i) and (ii) after Lemma 3·2 can be also proved to be true in the case of transform (1). The statement and proof of Theorem 4·1 are similar in the case of this transform and hence are omitted.

Corollary—If $F^{(\nu)}(x)$ be defined for positive real x as a special case of (12), then the following order relations are true:

$$| F^{(\nu)}(x) | = o([x^{-\nu}]) \; (x \to \infty, \nu = 1, 2, \ldots)$$

$$| F^{(\nu)}(x) | = o([x^{-\nu-1}]) \; (x \to 0+, \nu = 1, 2, \ldots)$$

$$(0 < x < 1).$$

The proof of these results is similar to that followed in Mishra (1972, p. 592) and Pandey (1972, p. 87) and hence is omitted.

Remark: The transform (3) also can be discussed with the same method as that employed in the development of (2) with the conditions (*A*) and (*B*) to be changed by putting $b = 0$, $a = \frac{1}{2} - m - k$ and $c = 2m$.

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