ON CLOSURES OF SUBMODULES

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It is shown that if a submodule of a module contains the singular submodule then its closure is closed. An alternative characterization of the closure has been provided and the well known property of non-singularity of a regular ring has been shown as an immediate consequence.

1. Introduction

Closures of submodules of a module were studied by Goldie (1960) and Johnson and Wong (1961). Goldie (1964) gave an explicit account of closures of submodules of a module. Ming (1969) gave very remarkable results on closures of submodules of a module and applied them to deduce some results on injectivity and to characterize self-injectivity of a ring R.

In the present note we derive a condition for the closure of submodule to be closed which generalizes a result of Faith (1967, p. 61). It is shown that the semi-simplicity of the ring R implies the nonsingularity of every R-module M (Proposition 3·6) and this provides a corollary that a nonzero module over a semi-simple ring with unit contains a regular element (Corollary 3·7). An alternative characterization of the closure of a submodule has also been provided (Theorem 4·1), and finally the well-known property of nonsingularity of a regular ring has been shown to follow as an immediate consequence (Corollary 4·4).

2. Notations and Preliminaries

$M \triangle N$ will mean that M is an essential extension of N. The injective hull of a module M is denoted by $\hat{M}$. Let M be an R-module. The closure of a submodule N of M as defined by Johnson and Wong (1961) and Goldie (1964) is

$$Cl_M(N) = \{m \in M / R \triangle (N : m)\}.$$

$Cl_M(0)$, the singular submodule of M is also denoted by $Z(M)$. if $Cl_M(0)=0$ then M is called a nonsingular module. Goldie (1964) and Armendariz (1970) call it as torsionfree module. A submodule N is closed in M if $M \supseteq P \triangle N \Rightarrow P = N$.

Here we have to notice a very important fact that the closure defined above is just a particular case of the closure defined by Goldie (1967) and can be said to be a closure of N with respect to the topological set **F** of large ideals (since (i) **F** is filtered, and (ii) $a \in R, F \in \mathbf{F} \Rightarrow (F : a) \in \mathbf{F}$).

Thus $Cl(N)$ defined above can precisely be recognized as $Cl\,\mathbf{F}(N)$, $Z(M)$ as $Z\,\mathbf{F}(M)$ and $Z(R)$ as $Z_{\mathbf{F}}(R)$; and so the singular submodule and the singular ideal considered by Johnson (1951) and Gentile (1962) are actually **F**-singular submodule of M and **F**-singular ideal of R in the terminology of Goldie (1967) where **F** is a topological set of large ideals. $Cl(N)$ may specifically be denoted by $Cl_{\mathbf{F}}^{M}(N)$.

3. Properties of Closures

Lemma 3·1—If $N \supseteq Cl\,(0)$ then $Cl\ N$ is closed.

PROOF: From Goldie (1964) we have

$Cl(N) \triangle N + Cl\,(0)$.

Since $N \supseteq Cl(0)$, $Cl(N) \triangle N$.

As in Lemma 2·2 of Goldie (1964) replacing N by $Cl(N)$ we get

$Cl\ Cl(N) \triangle Cl(N) \triangle N$

$\Rightarrow Cl\ Cl(N) \subseteq Cl(N)$

$\Rightarrow Cl\ Cl(N) = Cl(N)$

$\Rightarrow Cl\,(N)$ is closed.

As a consequence we derive an interesting result in the following:

Corollary 3·2—Let N be a submodule of a quasi-injective R-module M. Then $Cl\ N$ is quasi-injective if any one of the following holds:

(1) $N \supseteq Cl_M\,(0)$

(2) $Cl_R\,(0) = 0$.

PROOF: $N \supseteq Cl(0) \Rightarrow Cl(N)$ is closed (by the above Lemma)

$\Rightarrow Cl(N)$ is direct summand and hence quasi-injective by Harada (1965, Proposition 1·5 and its corollary).

(2) Ming (1969, Theo. 4) and the above result of Harada.

Corollary 3·3 — $Cl\ Cl(N)$ is closed for every submodule $N \subseteq M$ since $Cl(N) \supseteq Cl(0)$. In particular $Cl\ Cl(0)$ is closed.

This provides a quick proof of lemma 1 in Ming (1969).

Corollary 3·4—The submodules $N \subseteq M$ with the property that $N \supseteq Z(M)$ are contained in the unique closed essential extension $Cl(N)$.

This provides a general out-look to Faith (1967, Prop. 7, p. 61).

Definition—An element m in R-module M will be regular iff $(0 : m)=0$. If all the elements of a module are regular, the module is said to be regular. It is readily transparent that an element in R_R is regular iff it is not a zero-divisor in the ring R.

Remark 3·5—From Ming (1969, Prop. 7) it follows that R is nonsingular iff any nonsingular R-module M contains a regular element.

Proposition 3·6—If R is semi-simple with unit, then every R-module is nonsingular.

PROOF: Take an R-module M.

Suppose $Z(M) \neq 0$, then for any $0 \neq x \in Z'(M)$ we have $R \triangle (0 : x)$ so that $(0 : x) \neq 0$.

Now, since $(0 : x)$ is a direct summand of R, $(0 : x) = R$, and so

$1 \in (0 : x)$

$\Rightarrow x=0$, a contradiction.

Therefore, $Z(M)=0$.

Corollary 3·7—Every module over a semi-simple ring contains a regular element.

PROOF: By the Proposition, M is nonsingular and the corollary follows from Remark 3·5.

Corollary 3·8—If M is a regular module over a ring R then R is nonsingular.

PROOF: A regular module is obviously nonsingular and the result follows from Remark 3·5.

From Goldie (1964) we know that the closure operation is distributive over intersection. However, it does not always distribute over direct sum of submodules of M. For example, if $Z(M) \neq 0$ and A, B are complementary submodules of M, then

$$Cl(A \oplus B) \neq Cl(A) \oplus Cl(B).$$

Hence two closed submodules of M need not always be free. Here we observe: that if M be a module over a semi-simple ring (with unit) then for any free pair of submodules $A, B \subseteq M$, we have

$$Cl(A \oplus B) = Cl(A) \oplus Cl(B).$$

Since for any submodule N, $Cl\,N = \{ x \in M/R \triangle (N : x) \}$

and in this situation $R \triangle (N : x) \Rightarrow (N : x) = R \Rightarrow Rx \subseteq N$.

Since R has unit $x \in N$ and so $Cl\,N=N$. Whether it is true for a more general class of rings is yet to be seen.

Remark 3·9—It follows trivially from the above that in a semi-simple ring closure distributes over direct sums of pair of ideals.

Remark 3·10—From Armendariz (1970, Theorem 1) and Matlis (1958, Theorem 1·3) we deduce that over a ring R the direct sum of nonsingular injective modules is injective iff every nonsingular module $M = C \oplus N$ where C is a maximal injective one and the complementary summand N does not have any injective submodule.

Proposition 3·11—If $M = \oplus M_i$, then M is nonsingular iff each M_i is nonsingular.

PROOF: It follows from Sandomierski (1967, Prop. 2·4).

4. ALTERNATIVE FORMULATION

In the following theorem we provide an alternative characterisation of the closure of a submodule A of a module M:

Theorem 4·1 —Let $A' = \{ x \in M/(0 : \alpha) \subset (A : \alpha x)$ for every $\alpha \ (\neq 0) \in R \}$; and $B = \{ x \in M/R \vartriangle (A : x)\}$.

Then $A' = B$.

PROOF: Take $x \in A'$. Suppose $x \notin B$. Then for some $\alpha \neq 0$,

$R \alpha \cap (A : x) = 0.$

So, $\beta \alpha \in (A : x) \Rightarrow \beta \alpha = 0.$

i.e. $\beta \alpha x \in A \Rightarrow \beta \alpha = 0$

i.e. $\beta \in (A : \alpha x) \Rightarrow \beta \in (0 : \alpha)$

$\Rightarrow (A : \alpha x) \subseteq (0 : \alpha)$

$\Rightarrow x \notin A'$, a contradiction.

Therefore, $x \in A' \Rightarrow x \in B$.

Conversely, take $x \in B$. Then $R \vartriangle (A : x)$, and for $\alpha \neq 0$, $R \alpha \cap (A : x) \neq 0$

$\Rightarrow$ there exists $\beta \alpha \neq 0$, $\beta \alpha x \in A$

$\Rightarrow \beta \in (A : \alpha x) - (0 : \alpha)$

$\Rightarrow (0 : \alpha) \subset (A : \alpha x)$

$\Rightarrow x \in A'.$

Hence $A' = B$.

Now writing $Cl(A)$ for A' we get the desired formulation of the closure.

Corollary 4·2—$x \notin Cl(0) \Leftrightarrow (0 : \alpha) = (0 : \alpha x)$ for some $\alpha \ (\neq 0) \in R$.

Corollary 4·3—$x \in Cl(0)$ iff for every $\alpha \ (\neq 0) \in R$

$(0 : \alpha) \subset (0 : \alpha x)$.

Corollary 4·4 — A regular ring is nonsingular.

PROOF: Suppose R is regular and $x\ (\neq 0) \in Z(R)$. Then $x = x\,x'\,x$ for some $x' \in R$.

$\Rightarrow x'\, x \neq 0.$

Now, $x'\, x \in Cl_R(0)$, $x(\neq 0) \in R$

$\Rightarrow (0 : x) \subset (0 : xx'\, x) = (0 : x)$, a contradiction.

Hence $Z(R) = 0$.

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