ON BOCHNER CURVATURE TENSOR AND ITS APPLICATIONS

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Some properties of Bochner curvature tensor have been studied in the first section. Next we have considered an C-umbilical hypersurface of parallel Bochner curvature tensor.

INTRODUCTION

Let us consider $2m$-dimensional Kähler manifold V_{2m} with the structure (F, G) satisfying

$$\underline{\underline{x}} + x = 0 \qquad \ldots(1\cdot1a)$$

for arbitrary vector field x, where

$$\underline{x} \overset{def}{=} F(x) \qquad \ldots(1\cdot1b)$$

$$G(\underline{x}, \underline{y}) = G(x, y) \qquad \ldots(1\cdot2)$$

$$(E_x F)(y) = 0 \qquad \ldots(1\cdot3)$$

where E is the Riemannian connexion in V_{2m} and these equations will hold always for arbitrary vector fields x, y, z, u, v, etc. in V_{2m}. Let us put

$$'F(x, y) \overset{def}{=} G(\underline{x}, y). \qquad \ldots(1\cdot4)$$

Then $'F$ is skew-symmetric and hybrid in both the slots. Let K^* be the curvature tensor of E. Then in V_{2m}, we have

$$(a)\ K^*(x, y, \underline{z}) = \underline{K^*(x, y, z)}, \quad (b)\ K^*(\underline{x}, \underline{y}, z) = K^*(x, y, z) \qquad \ldots(1\cdot5)$$

(*a*) $^*Ric(\underline{x}, \underline{y}) = {}^*Ric(x, y)$, (*b*) $^*Ric(\underline{x}, y) + {}^*Ric(x, \underline{y}) = 0$, where $^*Ric$ is the Ricci tensor in V_{2m} and let ...(1·6)

$$(a)\ {}^*Ric(x, y) \overset{def}{=} G(r^*(x), y), \quad (b)\ r^*(\underline{x}) = \underline{r^*(x)}. \qquad \ldots(1\cdot7)$$

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Let us define the vector-valued trilinear functions L, M and C of type (1, 3) given by

$$\left.\begin{aligned} &(a)\ L(x, y, z) = {}^*Ric\,(x, y)_z + {}^*Ric\,(y, z)x + {}^*Ric\,(z, x)y \\ &(b)\ M(x, y, z) = {}^*r(z)\, G(x, y) + {}^*r(x)\, G(y, z) + {}^*r(y)\, G(z, x) \\ &(c)\ C(x, y, z) = G(x, y)z + G(y, z)x + G(z, x)y \end{aligned}\right\} \quad \ldots(1{\cdot}8)$$

which are symmetric in x, y, z.

The Bochner curvature tensor B^* in V_{2m} (Tachibana 1967) in view of (1·8) is given by

$$\begin{aligned} B^*(x, y, z) = K^*(x, y, z) &+ \frac{1}{2(m+2)}\,[\, L(\underline{x}, y, \underline{z}) - L(x, \underline{y}, \underline{z}) \\ &+ M(\underline{x}, y, \underline{z}) - M(x, \underline{y}, \underline{z})\,] \\ &- \frac{R^*}{4(m+1)(m+2)}\,[\, C(\underline{x}, y, \underline{z}) - C(x, \underline{y}, \underline{z})\,] \end{aligned} \quad \ldots(1{\cdot}9)$$

where R^* is the scalar curvature in V_{2m}. The tensor Q is said to be pure in all the three slots if

$$Q(\underline{x}, y, z) = Q(x, \underline{y}, z) = Q(x, y, \underline{z})$$

where Q is any of the tensors of type (1, 3). Now we have

Theorem 1·1—The Bochner curvature tensor coincides with the curvature tensor K^* iff L, M, and C are pure simultaneously in the first two slots.

Again we have

Theorem 1·2—The B^* coincides with the curvature tensor K^* if L, M are pure in the first two slots and $R^*=0$.

For an Einstein manifold V_{2m}, we have

$${}^*Ric\ (x, y) = \frac{R^*}{2m}\ G\,(x, y). \quad \ldots(1{\cdot}10)$$

Consequently the tensors L and M are equivalent and we have

$$L\,(x, y, z) = M\,(x, y, z) = R^*\ C\,(x, y, z)/2m \quad \ldots(1{\cdot}11)$$

Now for an Einstein manifold from (1·11) the Bochner curvature tensor B^* is ${}^*B^*$ given by

$${}^*B^*\,(x, y, z) = K^*\,(x, y, z) + \frac{R^*}{2m\,(m+1)}\ [C\,(\underline{x}, y, \underline{z}) - C\,(x, \underline{y}, \underline{z})] \quad \ldots(1{\cdot}12)$$

which is called Tachibana concircular curvature tensor (Rathore and Mishra 1973) and we have the following theorem.

Theorem 1·3—The Tachibana concircular curvature tensor coincides with the curvature tensor iff either $R^* = 0$ or C is pure in the first two slots.

Now with the help of (1·12) and (1·10), we can easily derive the following theorem due to Tachibana (1967).

Theorem 1·4—The Bochner curvature tensor coincides with Tachibana tensor $*B*$ iff V_{2m} is an Einstein manifold.

The manifold V_{2m} is said to be Bochner symmetric if

$$(E_v B^*)(x, y, z) = 0. \qquad \ldots(1.13)$$

On the other hand, we have considered an odd dimensional real differentiable manifold V_n (say $n = 2m - 1$). Let there exist a vector valued linear function f, a C^∞-vector field T and a C^∞ 1-form A satisfying

$$\overline{\overline{X}} + X = A(X)\, T \qquad \ldots(1\cdot14a)$$

where

$$\overline{X} \overset{def}{=} f(X) \qquad \ldots(1\cdot14b)$$

for arbitrary vector fields X, Y, Z, U, V, W etc in V_n. Then the manifold V_n is said to have an almost contact structure (f, T, A) and V_n is called an almost contact manifold. From (1·14a), we have

(*a*) $\overline{T}=0$, (*b*) $A(T)=1$, (*c*) $A(\overline{X})=0$, (*d*) the rank of matrix $((f))$ is $(n\text{-}1)$.

An almost contact manifold in which a metric tensor g satisfying ...(1·15)

$$g(\overline{X}, \overline{Y}) = g(X, Y) - A(X)\, A(Y), \qquad \ldots(1\cdot16)$$

has been introduced, is called an almost contact metric manifold. From (1·16) and (1·15b), we have

$$g(X, T) = A(X). \qquad \ldots(1\cdot17)$$

Let us put

$$'f(X, Y) = g(\overline{X}, Y).$$

Then $'f$ is skew-symmetric. An almost contact metric manifold in which

$$2\,'f(X, Y) = (D_X A)(Y) - (D_Y A)(X) \qquad \ldots(1\cdot18)$$

where D being Riemannian connexion, is satisfied, is called an almost Sasakian manifold. An almost Sasakian manifold in which

$$(D_X A)(Y) + (D_Y A)(X) = 0. \qquad \ldots(1\cdot19)$$

then the manifold is called K-contact Riemannian manifold. If in the K-contact Riemannian manifold

$$(D_X f)(Y) = A(Y)\, X - g(X, Y)\, T$$

is satisfied, then the manifold is called a Sasakian manifold. Thus in a Sasakian manifold, we have

$$(D_Z\, 'f)(X, Y) = 'K(X, Y, Z, T) = A(X)\, g(Y, Z) - A(Y)\, g(X, Z) \qquad \ldots(1\cdot20a)$$

$$'K(X, Y, Z, U) \overset{def}{=} g(K(X, Y, Z), U) \qquad \ldots(1\cdot20b)$$

$$Ric(X, T) = (n - 1)\, A(X) \qquad \ldots(1\cdot21)$$

$$K(T, Y, Z) = g(Y, Z)\, T - A(Z)\, Y \qquad \ldots(1\cdot22)$$

where K is the curvature tensor of V_n with respect to D and Ric is the Ricci tensor.

Let V_{2m-1} be the hypersurface of V_{2m} and let

$b : V_{2m-1} \rightarrow V_{2m}$,

be the inclusion map. The differential db of the map b will be denoted by B, so that to a vector X in V_{2m-1} there corresponds a vector BX in V_{2m}. If g is the induced metric tensor in V_{2m-1}, we have

$$(G\ (BX, BY))\ ob = g(X, Y). \qquad \ldots(1{\cdot}23)$$

Let N be the orthogonal unit normal vector field to V_{2m}. Then we have

$$(a)\ (G(BX,\ N))\ ob = 0, \qquad (b)\ G(N\ N) = 1. \qquad \ldots(1{\cdot}24)$$

If D is the induced Riemannian connexion in V_{2m-1}, then we have the following equations (Mishra 1972):

$$E_{BX}\, BY = BD_X\, Y + H(X, Y)\ N \quad \text{(Gauss equations)} \qquad \ldots(1{\cdot}25)$$

where H is symmetric in both the slots:

$$(a)\ E_{BX}\ N = -B\ 'H(X) \qquad \text{(Weingarten equations)},$$

$$(b)\ g('H(X), Y) = H(X, Y), \qquad \ldots(1{\cdot}26)$$

$$('K^*\ (BX, BY, BZ, BU))\ ob = 'K(X, Y, Z, U) - H(X, U)\ H(Y, Z) + H(X, Z)\ H(Y, U) \qquad \ldots(1{\cdot}27)$$

$$K^*\ (BX, BY, N) = -B(D_X\ 'H)\ (Y) + B(D_Y\ 'H)(X). \qquad \ldots(1{\cdot}28)$$

From (1·27), we have

$$^*Ric\ (BY, BZ) = Ric\ (Y, Z) - (C_1^1\ 'H)\ H(Y, Z) + H('H(Y), Z), \qquad \ldots(1{\cdot}29)$$

$$r^*\ (BY) = Br\ (Y) - (C_1^1 'H)\ B\ 'H(Y) + B\ 'H('H(Y)). \qquad \ldots(1{\cdot}30)$$

The conditions that V_{2m-1} be an almost contact metric hypersurface with the structure (f, T, A) in the Kähler manifold V_{2m} are (Mishra 1972)

$$F\,(BX) = B\overline{X} + A(X)\ N, \qquad \ldots(1{\cdot}31)$$

$$F\,(N) = -BT, \qquad \ldots(1{\cdot}32)$$

$$(a)\ D_X\overline{Y} + H(X, Y)\ T = \overline{D_X Y} + A(Y)\ 'H(X) \qquad \ldots(1{\cdot}33)$$

$$(b)\ H(X, \overline{Y}) + (D_X A)\,(Y) = 0.$$

In a Kähler manifold the induced structure (f, T, A) is Sasakian iff

$$H(X, Y) = g\ (X, Y) + dA(X)\ A(Y), \qquad \ldots(1{\cdot}34a)$$

where d given by

$$H(T, T) - 1 = d \qquad \ldots(1{\cdot}34b)$$

is the scalar field in V_{2m-1} and is related to the mean curvature by

$$d + 2m - 1 = (C_1^1\ 'H). \qquad \ldots(1{\cdot}34c)$$

Such a hypersurface is called C-umbilical hypersurface by Tashiro and Tachibana (1963). The scalar field d is covariant constant (Yamaguchi 1969).

2. Hypersurface of Kähler Manifold

Theorem 2·1—Let V_{2m-1} be an C-umbilical hypersurface of Kähler manifold V_{2m}. Then we have

$$(D_X H)(Y, Z) + (D_Y H)(Z, X) + (D_Z H)(X, Y) = 0. \quad ...(2\cdot 1a)$$

Proof: From (1·34), we have

$$(D_Z H)(X, Y) = d\, A(Y)(D_Z A)(X) + A(X)(D_Z A)(Y). \quad ...(2\cdot 1b)$$

Writing two other equations by cyclic permutation of X, Y, Z and adding and making use of (1·19), we get (2·1).

Corollary 2·1—Let V_{2m-1} be an C-umbilical hypersurface of Kähler manifold V_{2m}. Then the second fundamental form H is parallel in the direction of the C^∞ vector field T:

$$(D_T H)(X, Y) = 0.$$

Proof: From (1·14) in V_{2m-1}, we have

$$(D_X A)(T) = 0 \Rightarrow (D_T A)(X) = 0.$$

Putting T for Z in (2·1b) and using the last result, we have the statement.

Theorem 2·2—Let V_{2m-1} be an C-umbilical hypersurface of Bochner symmetric Kählerian manifold V_{2m}. Then V_{2m-1} is an C-Einstein manifold, that is,

$$Ric\,(Y, Z) = (\alpha + 2m-2)g(Y, Z) - \alpha A(Y)A(Z) \quad ...(2\cdot 2a)$$

where

$$\alpha \overset{def}{=} \frac{d_1}{2m+2}, \quad ...(2\cdot 2b)$$

$$d_1 \overset{def}{=} R + \{d(2m-3) - (d + 2m-2)\}(2m+1) + 4\,(m-1) - d\,(2m-3)\,(2m+5). \quad ...(2\cdot 2c)$$

Proof: Since for Bochner symmetric Kählerian manifold V_{2m}, we have (1·13) which implies

$$(E_{BV} B^*)\,(BX, BY, BZ) = 0.$$

Consequently, we have

$$((E_{BW} E_{BV} B^*) - (E_{BV} E_{BW} B^*) - (E_{[BW,\, BV]} B^*))\,(BX, BY, BZ) = 0.$$

Applying Ricci identity in this equation, we have

$$('K^*(BW, BV, B^*(BX, BY, BZ), BU))\, ob - ('B^*(K^*(BW, BV, BX), BY, BZ, BU))\, ob$$
$$- ('B^*(BX, K^*(BW, BV, BY), BZ, BU))\, ob$$
$$- ('B^*(BX, BY, K^*(BW, BV, BZ), BU))\, ob = 0 \quad ...(2\cdot 3a)$$

where

$$'B^*(BX, BV, BZ, BU) \overset{def}{=} G(B^*(BX, BY, BZ), BU). \quad ...(2\cdot 3b)$$

Substituting B^* from (1·9) in (2·3a), we get

$$'K^*(BW, BV, K^*(BX, BY, BZ), BU) - 'K^*(K^*(BW, BV, BX), BY, BZ, BU)$$
$$- 'K^*(BX, K^*(BW, BV, BY), BZ, BU) - 'K^*(BX, BY, K^*(BW, BV, BZ), BU)$$
$$+ \frac{1}{2(m+2)} [\{ ^*Ric(K^*(BW, BV, BY), BZ) + {}^*Ric(BY, K^*(BW, BV, BZ)) \} G(BX, BU)$$
$$- \{ ^*Ric(K^*(BW, BV, BX), BZ) + {}^*Ric(BX, K^*(BW, BV, BZ)) \} G(BY, BU)$$
$$+ \{ 'K^*(BW, BV, r^*(BY), BU) - {}^*Ric(K^*(BW, BV, BY), BU)) \} G(BX, BZ)$$
$$- \{ 'K^*(BW, BV, r^*(BX), BU) - {}^*Ric(K^*(BW, BV, BX), BU)) \} G(BY, BZ)$$
$$+ \{ ^*Ric(K^*(BW, BV, F(BY), BZ) + {}^*Ric(F(BY), K^*(BW, BV, BZ)) \} 'F(BX, BU)$$
$$- \{ ^*Ric(K^*(BW, BV, F(BX)), BZ) + {}^*Ric(F(BX), K^*(BW, BV, BZ)) \} 'F(BY, BU)$$
$$+ \{ 'K^*(BW, BV, r^*(F(BY)), BU) - {}^*Ric(K^*(BW, BV, F(BY)), BU) \} 'F(BX, BZ)$$
$$- \{ 'K^*(BW, BV, r^*(F(BX)), BU) - {}^*Ric(K^*(BW, BV, F(BX)), BU) \} 'F(BY, BZ)$$
$$+ 2 \{ 'K^*(BW, BV, r^*(F(BZ)), BU) - {}^*Ric(K^*(BW, BV, F(BZ)), BU) \} 'F(BX, BY)$$
$$- 2 \{ ^*Ric(K^*(BW, BV, F(BX)), BY) + {}^*Ric(F(BX), K^*(BW, BV, BY)) \} 'F(BZ, BU)]$$
$$= 0. \qquad \ldots(2{\cdot}4a)$$

The computation of the above equation is very complicated and lengthy. Therefore, we give here only the indication by solving a single term that how equation (2·4a) is being tackled.

$$Ric(K^*(BW, BV, F(BX)), BY) = - {}^*Ric(K^*(BW, BV, BX), F(BY))$$
$$= - {}^*Ric(K^*(BW, BV, BX), BU) - {}^*Ric(K^*(BW, BV, BX), N)A(Y). \qquad \ldots(2{\cdot}4b)$$

By making use of $^*Ric(BY, N) = 0$, and (1·27)–(1·29), we have

$$= - {}^*Ric(BK(W, V, X), B\bar{Y}) - {}^*Ric(B'H(W), B\bar{Y})H(V, X) + {}^*Ric(B'H(V), B\bar{Y})H(W, X),$$
$$= - Ric(K(W, V, X), \bar{Y}) + (C_1^1\, 'H)H(K(W, V, X), \bar{Y}) - H('H(K(W, V, X), \bar{Y})$$
$$+ \{ Ric('H(W), \bar{Y}) - (C_1^1\, 'H)H('H(W), \bar{Y}) + H('H('H(W)), \bar{Y}) \} H(V, X)$$
$$\{ - Ric('H(V), \bar{Y}) + (C_1^1\, 'H)H('H(V), \bar{Y}) - H('H('H(V)), \bar{Y}) \} (HW, X).$$

Substituting from (1.23)–(1.34) in the other terms of equation (2.4*a*) then putting T for W and T for U and by making use of (1.14)–(1.22), we get

$$d'K(X, Y, Z, U) = d(1-d) \{ g(X, V)g(Y, Z) - g(Y, V)g(X, Z) \}$$
$$d^2 \{ (g(Y, Z)A(X) - g(X, Z)A(Y))A(V) + (g(X, V)A(Y) - g(V, Y)A(X))A(Z) \}$$
$$+ \frac{d}{2(m+2)} [\{ Ric(V, Y)A(X) - Ric(V, X)A(Y) \} A(Z) +$$

(*equation contd. next page*)

$$+d(2m-2)-(d+2m-2))\ \{(g(V,Y)A(X)-g(V,X)A(Y))A(Z)$$
$$-g(V,Y)g(X,Z)+g(V,X)g(Y,Z)-'f(Y,V)'f(X,Z)+'f(X,V)'f(Y,Z)$$
$$-2'f(Z,V)'f(X,Y)\}-Ric\,(V,Y)g(X,Z)+Ric\,(V,X)g(Y,Z)$$
$$+d(2m-3)A(V)\ \{g(X,Z)A(Y)-g(Y,Z)A(X)\}$$
$$-Ric\,(\bar{Y},V)'f(X,Z)-Ric\,(\bar{X},V)'f(Y,Z)-2Ric\,(\bar{Z},V)'f(X,Y)]. \qquad ...(2.5)$$

From the above equation, we are able to see that $d=0$, which is impossible. Hence contracting (2.5) and using (1.14)–(1.16) and (1.21), we get (2.2).

Corollary 2.2—Let V_{2m-1} be an C-umbilical hypersurface of Bochner symmetric Kähler manifold V_{2m}. Then V_{2m-1} is almost locally C-Fubinian manifold, that is

$$'K(X,Y,Z,V)=(1+d_2-d)\{g(X,V)g(Y,Z)-g(X,Z)g(V,Y)\}$$
$$d_2\{'f(X,V)'f(Y,Z)-'f(X,Z)'f(Y,V)-2'f(X,Y)'f(Z,V)\}$$
$$-(d_2-d)\{(g(Y,Z)A(V)-g(Y,V)A(Z))A(X)$$
$$-(g(X,Z)A(V)-g(X,V)A(Z))A(Y)\} \qquad ...(2.6a)$$

where

$$2(m+2)d_2\overset{def}{=}\alpha+d\,(2m-3). \qquad ...(2.6b)$$

PROOF: Substituting from (2.2*a*) in (2.5), we get (2.6*a*), which we shall call almost locally C-Fubinian manifold.

Corollary 2.3–An C-umbilical hypersurface of Bochner symmetric Kähler manifold is of constant scalar curvature tensor R given by

$$2R=2(m^2-1)(2m-1)+\{d(2m-3)-(d+2m-2)\}\,(2m+1)(m-1)$$
$$+\,4(m-1)^2-d(2m-3)(2m+5)(m-1). \qquad ...(2.7)$$

PROOF: From (2.2*a*), we have

$$(2m+2)r(Y)=(d_1+4(m^2-1))Y-d_1\,A(Y)\,T.$$

Contracting the above equation, we get

$$(2m+2)R=\{d_1+4(m^2-1)\}\,(2m-1)-d_1.$$

Substituting d_1 from (2.2) in the above equation, we get (2.7).

Inspite of the proof of Corollary 2.3 given by the present author, the corollary can be stated directly from the theorems of Matsumoto and Tano (1973) and Yamaguchi (1969).

Theorem 2.3 (Matsumoto and Tano 1973)—Let V_{2m} be a Kählerian space with parallel Bochner curvature tensor. Then V_{2m} is a locally symmetric or a space with vanishing Bochner curvature tensor.

Theorem 2.4 (Yamaguchi 1969)—Let V_{2m-1} be an C-umbilical hypersurface of a Kählerian manifold V_{2m} with vanishing Bochner curvature tensor. Then V_{2m-1} is locally C-Fubinian (S. Yamaguchi).

Now from Theorem 2.3 and Theorem 2.4, we have a statement.

Theorem 2.5—Let V_{2m-1} be an C-umbilical hypersurface of Kählerian manifold V_{2m} with parallel Bochner curvature tensor. Then V_{2m-1} is locally C-Fubinian.

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REFERENCES

Matsumoto, M., and Tano, S. (1973). Kählerian spaces with parallel or vanishing Bochner curvature tensor. *Tensor N.S.*, **27**, 291-94.

Mishra, R. S. (1972). Almost complex and almost contact submanifolds. *Tensor N.S.*, **25**, 429-33.

Rathore, M. P. S., and Mishra, R. S. (1973). Properties of Tachibana concircular curvature tensor. *Indian J. pure appl. Math.*, **4**, 568-75.

Tachibana, S. (1967). On the Bochner curvature tensor. *Natural Sci. Rep. Ochanomizu Univ.*, **18**, 15-19.

Tashiro, Y., and Tachibana, S. (1963). Fubinian and C-Fubinian manifold. *Kodai Math. Sem. Rep.*, **15**, 170-83.

Yano, K. (1965). Differential Geometry on Complex and Almost Complex Spaces, Pergamon Press.

Yamaguchi, S. (1969). On a C-umbilical hypersurface of a Kähler manifold with vanishing Bochner curvature tensor. *Tensor, N.S.*, **20**, 95-99.