ON $| N_p, \gamma, \alpha |_k$ SUMMABILITY OF INFINITE SERIES

by SARFARAZ UMAR *and* HUZOOR H. KHAN, *Department of Mathematics, Aligarh University, Aligarh*

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The well-known definition of Abel summability $| A |$ is due to Whittaker (1930-31). Further extension of summability $| A |$ was obtained by Flett (1957). We have extended the definition of absolute Nörlund summability and prove a theorem which gives sufficient conditions for the series Σu_n to be summable $| A, \gamma |_k$ whenever it is summable by our new method $| N_p, \gamma, \alpha |_k$. Clearly summability $| N_p, 0, \alpha |_1$ is the same as summability $| N_p |$ and summability $| N_p, 0, 1 |_k$ reduces to the summability $| N_p |_k$.

§1. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + \dots + p_n \neq 0; \; p_{-1} = P_{-1} = 0.$$

We call t_n the N_p or Nörlund transform (or mean) of an infinite series Σu_n (with the sequence of partial sums $\{U_n\}$), where

$$t_n = \sum_{v=0}^{n} \frac{p_{n-v}}{P_n} u_v = \sum_{v=0}^{n} \frac{p_{n-v}\, U_v}{P_n}. \qquad \dots(1.1)$$

The series Σu_n, or the sequence $\{U_n\}$ is said to be summable by Nörlund mean, generated by the sequence $\{p_n\}$ or simply summable N_p, if $\lim_{n \to \infty} t_n$ exists.

The conditions of regularity for the N_p-transform (1.1), are

$$\text{(i)} \qquad \lim_{n \to \infty} \frac{p_n}{P_n} = 0 \qquad \dots(1.2)$$

and

$$\text{(ii)} \qquad \sum_{k=0}^{n} | p_k | \leqslant C \, | P_n | \qquad \dots(1.3)$$

where C is a finite positive constant.

We might also note that if $\{p_n\}$ is real and non-negative, then condition (1.2) is satisfied automatically, and if in addition $\{p_n\}$ is non-increasing, condition (1.1) is also satisfied.

The series Σu_n, or the sequence $\{U_n\}$, is said to be absolutely summable by the Nörlund mean defined by the sequence $\{p_n\}$ or simply summable $| N_p |$, if

$$\sum_{n=1}^{\infty} | t_n - t_{n+1} | \leqslant C^*.$$

We define the series Σu_n, or the sequence $\{U_n\}$ to be summable $| N_p, \gamma, \alpha |_k$ if

$$\sum_{n=1}^{\infty} \left| \frac{P_n}{p_n} \right|^{\alpha(k\gamma+k-1)} | t_n - t_{n-1} |^k < C$$

where $k \geqslant 1, \gamma \geqslant 0$ and α is real.

Clearly summability $| N_p, 0, \alpha |_1$ is the same as summability $| N_p |$ and summability $| N_p, 0, 1 |_k$ reduces to the summability $| N_p |_k$.

Following the definition of Whittaker (1930-31) we say that the series Σu_n, or the sequence $\{U_n\}$ is absolutely Abel summable $| A |$ if $\sum\limits_{n=0}^{\infty} u_n x^n$ is convergent for $0 \leqslant x < 1$ and its sum function, $f(x)$, is of bounded variation in (0, 1), that is, $\int_0^1 | f'(x) | \, dx$ exists.

Further extension of the summability $| A |$ was obtained by Flett (1957). Thus according to him the series Σu_n or the sequence $\{U_n\}$ is said to be summable $| A, \gamma |_k$ where $k \geqslant 1$ and γ a real number, if the series $\Sigma u_n x^n$ is convergent for $0 \leqslant x < 1$, and its sum function $f(x)$ satisfies the condition

$$\int_0^1 (1 - x)^{-k\gamma+k-1} | f'(x) |^k \, dx \leqslant C.$$

Clearly summability $| A, 0 |_1$ is the absolute Abel summability.

The object of this note is to extend the definition of absolute Nörlund summability and to prove a theorem which gives sufficient conditions for the series Σu_n, or the sequence $\{U_n\}$, to be summable $| A, \gamma |_k$ whenever it is summable by our new method.

§2. We prove the following theorem :

Theorem — Let N_p be a regular Nörlund transformation and

*Throughout C denotes an absolute constant, not necessarily the same at each occurrence.

$$\phi_n(x) = \sum_{k=0}^{n} p_k x^k \Big/ \sum_{k=0}^{\infty} P_k x^k.$$

If

(i) $\sum_{n=0}^{\infty} u_n$ is summable $| N_p, \gamma, \alpha |_k$ and

(ii) $$^*\sum_{n=1}^{\infty} \left| \frac{P_n}{p_n} \right|^{-((k\gamma/k-1)+1)\alpha} < C \text{ and}$$

(iii) $$\int_0^1 (1-x)^{-k\gamma+k-1} | \phi'(x) |^k dx \text{ is uniformly bounded,}$$

for $n \geqslant 0$,

then $\sum_{n=0}^{\infty} u_n$ is summable $| A, \gamma |_k$, $k \geqslant 1, \gamma \geqslant 0$ and $\alpha \geqslant 0$.

PROOF : Writing $P(x) = \sum_{n=0}^{\infty} p_n x^n$ and $R(x) = \sum_{n=0}^{\infty} P_n t_n x^n$,

we have clearly

$$R(x) = \sum_{n=0}^{\infty} P_n \cdot \frac{1}{p_n} \left(\sum_{v=0}^{n} p_{n-v} u_v \right) x^n = f(x) \cdot P(x).$$

Hence $f(x) = R(x)/P(x)$. Since N_p-transform is regular, we have

$$\lim_{n\to\infty} \frac{P_n}{P_{n-1}} = 1, \text{ so } P(x) \text{ has a radius of convergence } 1.$$

In view of the hypothesis of the theorem

$$\sum_{n=0}^{\infty} \left| \frac{P_n}{p_n} \right|^{\alpha(k\gamma+k-1)} | t_n - t_{n-1} |^k < \infty$$

which implies that

$$\sum_{n=0}^{\infty} | t_n - t_{n-1} | < \infty$$

*This condition is not required when $\gamma = 0$ and $k = 1$.

for

$$\sum_{n=1}^{\infty} | t_n - t_{n-1} | = \sum_{n=1}^{\infty} | t_n - t_{n-1} | \left| \frac{P_n}{p_n} \right|^{\alpha(k\gamma+k-1)/k} \left| \frac{P_n}{p_n} \right|^{-\alpha(k\gamma+k-1)/k}$$

$$\leqslant \left\{ \sum_{n=1}^{\infty} | t_n - t_{n-1} |^k \left| \frac{P_n}{p_n} \right|^{\alpha(k\gamma+k-1)} \right\}^{1/k}$$

$$\times \left\{ \sum_{n=1}^{\infty} \left| \frac{P_n}{p_n} \right|^{-\alpha k'(k\gamma+k-1)/k} \right\}^{1/k'}, \; 1/k + 1/k' = 1$$

$$\leqslant C \cdot \left\{ \sum_{n=1}^{\infty} \left| \frac{P_n}{p_n} \right|^{-\alpha(k\gamma+k-1)/k-1} \right\}^{1/k'}$$

$$= C \cdot \left\{ \sum_{n=1}^{\infty} \left| \frac{P_n}{p_n} \right|^{-((k\gamma/k-1)+1)\alpha} \right\}^{1/k'}, \; k > 1, \gamma = 0$$

$$\leqslant C.$$

Hence $t_n \to l$ (l finite), say, and thus $R(x)$ has a radius of convergence 1, i.e., it has the same radius of convergence as that of $P(x)$. Thus $f(x) = \Sigma u_n x^n$ converges for $0 \leqslant x < 1$.

Therefore

$$f'(x) = \frac{P(x) \cdot R'(x) - P'(x) R(x)}{P^2(x)}$$

$$= \frac{P(x) \cdot \sum_{n=0}^{\infty} nP_n t_n x^{n-1} - P'(x) \sum_{n=0}^{\infty} P_n t_n x^n}{P^2(x)}.$$

The numerator by Abel's transformation is

$$= P(x) \lim_{N\to\infty} \sum_{n=0}^{N} \left(\sum_{k=0}^{n} kP_k x^{k-1} \right)(t_n - t_{n+1}) + P(x) \lim_{N\to\infty} t_N \left(\sum_{k=0}^{N} kP_k x^{k-1} \right)$$

$$- P'(x) \lim_{M\to\infty} \sum_{n=0}^{M} \left(\sum_{k=0}^{n} P_k x^k \right)(t_n - t_{n-1}) - P'(x) \lim_{M\to\infty} t_M \left(\sum_{k=0}^{M} P_k x^k \right)$$

$$= P(x) \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} kP_k x^{k-1} \right)(t_n - t_{n+1}) + P(x)\, l \sum_{k=0}^{\infty} kP_k x^{k-1}$$

$$- P'(x) \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} P_k x^k \right)(t_n - t_{n+1}) - P'(x)\, l \sum_{k=0}^{\infty} P_k x^k$$

$$= P(x) \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} kP_k x^{k-1} \right) (t_n - t_{n+1}) + lP(x)\, P'(x)$$

$$- P'(x) \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} P_k x^k \right) (t_n - t_{n+1}) - lP'(x)\, P(x).$$

Hence

$$f'(x) = \frac{\{P(x) \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} kP_k x^{k-1} \right) (t_n - t_{n+1}) - P'(x) \sum_{k=0}^{\infty} \left(\sum_{k=0}^{n} P_k x^k \right) (t_n - t_{n+1})\}}{P^2(x)},$$

$$= \sum_{n=0}^{\infty} \frac{\left\{ P(x) \frac{d}{dx} (P_0 + P_1 + \ldots + P_n x^n) - P'(x)\, (P_0 + P_1 x + \ldots + P_n x^n) \right\} (t_n - t_{n+1})}{P^2(x)}$$

$$= \sum_{n=0}^{\infty} \phi_n'(x)\, (t_n - t_{n+1}),$$

so that

$$| f'(x) |^k \leqslant \left\{ \sum_{n=0}^{\infty} | \phi_n'(x) | \; | t_n - t_{n+1} | \right\}^k,$$

$$= \left\{ \sum_{n=0}^{\infty} | \phi_n'(x) | \; \left| \frac{P_n}{p_n} \right|^{\alpha(k\gamma+k-1)/k} | t_n - t_{n+1} | \left| \frac{P_n}{p_n} \right|^{-\alpha(k\gamma+k-1)/k} \right\}^k$$

Applying Hölder's inequality, we get

$$| f'(x) |^k \leqslant \left\{ \sum_{n=0}^{\infty} | \phi_n'(x) |^k \left| \frac{P_n}{p_n} \right|^{\alpha\{k\gamma+k-1\}} \right\}$$

$$\times \; | t_n - t_{n+1} |^k \left\{ \sum_{n=0}^{\infty} \left| \frac{P_n}{p_n} \right|^{-\alpha\{k(k\gamma+k-1)/k\}} \right\}^{k/k'},$$

$$\leqslant C \sum_{n=0}^{\infty} | \phi_n'(x) |^k \left| \frac{P_n}{p_n} \right|^{\alpha(k\gamma+k-1)} | t_n - t_{n+1} |^k.$$

Therefore, by hypothesis, we have

$$\int_0^1 (1-x)^{-k\gamma+k-1} \, | f'(x) |^k \, dx$$

$$\leqslant A \sum_{n=0}^{\infty} \int_0^1 (1-x)^{-k\gamma+k-1} \, | \phi_n'(x) |^k \, dx \left| \frac{P_n}{p_n} \right|^{\alpha(k\gamma+k-1)} | t_n - t_{n+1} |^k,$$

$$\leqslant C \sum_{n=0}^{\infty} \left| \frac{P_n}{p_n} \right|^{\alpha(k\gamma+k-1)} | t_n - t_{n+1} |^k,$$

$$< \infty \, .$$

This completes the proof of the theorem.

If we consider the special case $k = 1, \gamma = 0$, we get the following theorem of McFadden (1942).

Theorem — Let N_p be a regular Nörlund transformation. If Σu_n is summable $| N_p |$ and the sequence $\phi_n(x)$ is uniformly of bounded variation, then Σu_n is summable $| A |$.

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