COMPARATIVE STUDY OF TWO-DIMENSIONAL AND AXI-SYMMETRIC JET FLOWS OF POWER LAW LIQUID

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A comparative study on the two-dimensional and axi-symmetric jet flows of power law liquid issuing into a semi-infinite space is presented in this paper for different values of the flow behaviour index *n*. It brings about no significant error if the entire flow-region is divided into two parts, the zone of flow establishment where the flow develops to its full growth and the zone of established flow where the flow is self-preserving. In a two-dimensional jet flow the expressions for the velocity distributions and half widths of the jet are derived separately for both the regions. Expression for the half width of the potential core in the first region is also calculated. The same investigation is carried out in case of an axi-symmetric jet flow. Here the integrals of the boundary layer equations are solved numerically with the help of Simpson's one-third rule.

INTRODUCTION

Recently Hatta and Nozaki (1975) have investigated the two-dimensional and axi-symmetric jet flows of Newtonian liquid extensively. The experimental data are found to be in good agreement with the theoretical results. In this paper the problem has been extended to the power law liquid where Newtonian liquid is simply a particular case.

In a jet there are three distinct regions, of which the central region is a transition between the first, where the jet consists of a potential core bounded by two turbulent mixing layers, and the third where the jet flow is self-preserving. For the sake of simplicity in calculation the transition region is ignored, so that the jet is taken to grow as a free jet from the point where the initial mixing layers have grown sufficiently to have entrained the whole of the potential core. The first region is called the zone of establishment and the remaining one is called the zone of established flow.

TWO-DIMENSIONAL JET FLOW ISSUING FROM A FINITE WIDTH NOZZLE

A two-dimensional steady jet is issuing into a semi-infinite space from a finite width nozzle. Since inside of the jet is concerned, here the boundary layer appoximations are applicable to the equations of motion. To construct the equations of motion let the origin be taken at the centre of the nozzle exit, the x-axis along the jet centre-line and the y-axis perpendicular to it. For a laminar flow let u, v denote the velocity components while for a turbulent flow they represent the components of mean velocity.

Here the velocity gradient is everywhere negative and accordingly a positive shear stress τ_{xy} for a power law liquid has the form

$$\tau_{xy} = - K \left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y}$$

Hence for a power law liquid the corresponding equation of motion is

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - k \frac{\partial}{\partial y} \left(- \frac{\partial u}{\partial y} \right)^n, \; k = K/\rho. \qquad \ldots(1)$$

Integration of eqn. (1) gives the momentum integral equation

$$\frac{d}{dx} \int_0^\infty u^2 dy = 0 \qquad \ldots(2)$$

and also from Eq. (1) the energy integral equation becomes

$$\tfrac{1}{2} \frac{d}{dx} \int_0^\infty u^3 dy + k \int_0^\infty \left(- \frac{\partial u}{\partial y} \right)^{n+1} dy = 0. \qquad \ldots(3)$$

Let $2b_0$ denote the nozzle width and U_0 the uniform velocity at the nozzle exit. Thus on integration eqn. (2) assumes the form

$$\int_0^\infty u^2 dy = U_0^2 \, b_0. \qquad \ldots(4)$$

Now for the two aforesaid regions the solutions are given separately.

Zone of Established Flow

In the zone of established flow the centre-line velocity component U is lower than U_0 and there are no regions where the velocity components are constant. In this zone let us assume $2b$ be a width of the jet and the velocity component as

$$u = Uf(\eta) \qquad \ldots(5)$$

where

$$\eta = y/b.$$

For velocity profile $f(\eta)$ let us take the expression used by Hatta and Nozaki (1975) and which has proved most successful for Newtonian liquid.

$$f(\eta) = (1 - \eta)^3 (1 + 3\eta). \qquad \text{...(6)}$$

Using the relation (5) eqn. (4) reduces to

$$U^2 b \int_0^1 f^2 \, d\eta = U_0^2 \, b_0. \qquad \text{...(7)}$$

Considering the dimension of the material constant k in eqn. (1) we assume the relation

$$k = k' b_m^n \, U^{2-n} \qquad \text{...(8)}$$

where k' is independent of time and determined from the experiment, b_m is a half mean width of the jet defined by

$$U b_m = \int_0^b u \, dy. \qquad \text{...(9)}$$

On simplification eqns. (7) and (9) give respectively

$$b = \tfrac{7}{2} b_0 \left(\frac{U_0}{U} \right)^2 \qquad \text{...(10)}$$

and

$$b_m = \tfrac{2}{5} b. \qquad \text{...(11)}$$

Combining the relations (8) and (11) we get

$$k = k'(\tfrac{2}{5} b)^n \, U^{2-n}. \qquad \text{...(12)}$$

Eqn. (3) thus reduces to

$$\tfrac{1}{2} \frac{d}{dx} (U^3 b) \int_0^1 f^3 d\eta + k'(\tfrac{2}{5})^n \, U^3 \int_0^1 (- f')^{n+1} \, d\eta = 0. \qquad \text{...(13)}$$

The boundary condition for this differential equation is $U = U_0$ at $x = x_1$, where x_1 is the length of the zone of flow establishment.

Hence using the relation (10) the solution of the differential equation (13) is given by

$$\frac{U}{U_0} = \left[1 + \frac{2860 \, k'}{581 \, b_0} (\tfrac{2}{5})^n \, I(x - x_1) \right]^{-1/2} \qquad \text{...(14)}$$

giving a half width of the jet

$$\frac{b}{b_0} = \frac{7}{2} \left[1 + \frac{2860 \, k'}{581 \, b_0} (\tfrac{2}{5})^n \, I(x - x_1) \right] \qquad \text{...(15)}$$

where

$$I = \int_0^1 (-f')^{n+1}\, d\eta.$$

Zone of Flow Establishment

In the potential core of the zone of flow establishment the velocity component U_0 is uniform and outside the potential core it gradually diminishes from U_0 to O. Let $2b_c$ and $2b$ be the respective widths of the potential core and the jet. Therefore we can assume

$$u = U_0 f(\eta) \qquad \text{...(16)}$$

in $b_c < y < b$, where $\eta = (y - b_c)/(b - b_c)$ and $f(\eta)$ is defined by eqn. (6).

On integration momentum integral equation (2) becomes

$$\int_0^{b_c} U_0^2\, dy + \int_{b_c}^{b} u^2\, dy = U_0^2\, b_0. \qquad \text{...(17)}$$

Using transformation (16), eqn. (17) reduces to

$$U_0^2\, b_c + U_0^2 (b - b_c) \int_0^1 f^2 d\eta = U_0^2\, b_0 \qquad \text{...(18)}$$

For material constant k we assume as in eqn. (8)

$$k = k'(b_m - b_c)^n\, U_0^{2-n} \qquad \text{...(19)}$$

where a half mean width b_m is defined by

$$U_0 b_m = U_0 b_c + \int_{b_c}^{b} u\, dy. \qquad \text{...(20)}$$

Hence on simplifications eqns. (18) and (20) give respectively

$$b = \tfrac{7}{2}\, b_0 - \tfrac{5}{2}\, b_c \qquad \text{...(21)}$$

and

$$b_m - b_c = \tfrac{2}{5}\, (b - b_c). \qquad \text{...(22)}$$

The energy integral equation is given by

$$\tfrac{1}{2} \frac{d}{dx} (U_0^3\, b_c) + \tfrac{1}{2} \frac{d}{dx} \int_{b_c}^{b} u^3 dy + k \int_{b_c}^{b} \left(- \frac{\partial u}{\partial y} \right)^{n+1} dy = 0. \qquad \text{...(23)}$$

Combining eqns. (19), (22) and (23) we get

$$\frac{db_c}{dx} + \frac{715}{67} \left(\tfrac{2}{5}\right)^n Ik' = 0. \qquad \ldots(24)$$

Using (21) integration of eqn. (24) gives the half width of the potential core b_c and the half width of the jet b

$$b_c = b_0 - \frac{715}{67} \left(\tfrac{2}{5}\right)^n Ik'x \qquad \ldots(25)$$

and

$$b_c = b_0 + \frac{715}{67} \left(\tfrac{2}{5}\right)^{n-1} Ik'x \qquad \ldots(26)$$

respectively subject to the initial conditions

$$b_c = b = b_0 \text{ at } x = 0$$

where I is already defined.

Axi-symmetric Jet Flow Issuing From A Finite Radius Nozzle

An axi-symmetric steady jet is issuing into a semi-infinite space from a finite radius nozzle. We take the origin at the centre of the nozzle exit, the x-axis along the jet centre-line and the y-axis along the radial distance from the x-axis. The equation of motion in the x-direction is given by

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -k\frac{\partial}{y\partial y}\left[y\left(-\frac{\partial u}{\partial y}\right)^n\right]. \qquad \ldots(27)$$

since the velocity gradient is everywhere negative.

Here the momentum integral equation is

$$2\frac{d}{dx}\int_0^\infty u^2 y\, dy = 0 \qquad \ldots(28)$$

while the energy integral equation is

$$\tfrac{1}{2}\frac{d}{dx}\int_0^\infty u^3 y\, dy + k\int_0^\infty y\left(-\frac{\partial u}{\partial y}\right)^{n+1} dy = 0. \qquad \ldots(29)$$

Taking b_0 to be the radius of the nozzle exit and U_0 the uniform velocity at the nozzle exit integration of eqn. (28) gives

$$2 \int_0^\infty u^2 y \, dy = U_0^2 \, b_0^2. \qquad \ldots(30)$$

Zone of Established Flow

In this zone let b denote the radius of the jet and let the velocity component be

$$u = Uf(\eta) \qquad \ldots(31)$$

where

$$\eta = y/b \text{ and } f(\eta) = (1 - \eta)^4 \, (1 + 4\eta). \qquad \ldots(32)$$

This form of $f(\eta)$ is chosen by Hatta and Nozaki for axi-symmetric jet flows of Newtonian fluid and their calculations are found to tally very closely to the experimental results. For power law fluid the same expression for $f(\eta)$ is being applied here.

Thus eqn. (30) reduces to

$$2U^2 b^2 \int_0^1 \eta f^2 \, d\eta = U_0^2 \, b_0^2 \qquad \ldots(33)$$

while eqn. (29) becomes

$$\tfrac{1}{2} \frac{d}{dx} (U^3 b^2) \int_0^1 \eta f^3 \, d\eta + kb^2 \left(\frac{U}{b} \right)^{n+1} \int_0^1 \eta(-f')^{n+1} \, d\eta = 0. \qquad \ldots(34)$$

Similar to eqn. (8) we assume

$$k = k' b_m^n \, U^{2-n} \qquad \ldots(35)$$

where b_m, a mean radius of the jet, is defined by

$$b_m^2 \, U = 2 \int_0^\infty uy \, dy. \qquad \ldots(36)$$

Hence applying eqns. (35) and (36) we get from eqn. (34)

$$\frac{d}{dx} (U^3 b^2) + I_1 \, k' \, \frac{6188}{71} \, \frac{b}{7^{n/2}} \qquad \ldots(37)$$

with

$$I_1 = \int_0^1 \eta(-f')^{n+1} \, d\eta.$$

The initial condition to be used is $U = U_0$ at $x = x_1$, where x_1 is the length of the zone of flow establishment.

Integration of eqn. (37), by using eqn. (30), gives

$$\frac{U}{U_0} = \left[1 + \frac{6188\sqrt{11}}{2343} \frac{k'}{7^{(n-1)/2} b_0} I_1 (x - x_1) \right]^{-1} \qquad \ldots(38)$$

from which b, a radius of the jet and b_m, a mean radius of the jet are respectively reduced to

$$\frac{b}{b_0} = \frac{3\sqrt{77}}{7} \left[1 + \frac{6188\sqrt{11}}{2343} \frac{k'}{7^{(n-1)/2} b_0} I_1(x - x_1) \right] \qquad \ldots(39)$$

and

$$\frac{b_m}{b_0} = \frac{3\sqrt{11}}{7} \left[1 + \frac{6188\sqrt{11}}{2343} \frac{k'}{7^{(n-1)/2} b_0} I_1(x - x_1) \right] \qquad \ldots(40)$$

where I_1 is defined after eqn. (37).

Zone of Flow Establishment

In the zone of flow establishment eqn. (30) reduces to

$$2 \int_0^{b_c} U_0^2 y \, dy + 2 \int_{b_c}^{b} u^2 y \, dy = U_0^2 \, b_0^2 \qquad \ldots(41)$$

and energy integral eqn. (29) reduces to

$$\tfrac{1}{2} \frac{d}{dx} \left(\int_0^{b_c} U_0^3 y \, dy + \int_{b_c}^{b} u^3 y \, dy \right) + k \int_{b_c}^{b} y \left(- \frac{\partial u}{\partial y} \right)^{n+1} dy = 0 \qquad \ldots(42)$$

Similar to eqn. (19) we take

$$k = k'(b_m - b_c)^n \, U_0^{2-n} \qquad \ldots(43)$$

with a mean radius of the jet defined by

$$b_m^2 \, U_0 = b_c^2 \; U_0 + 2 \int_{b_c}^{b} uy \, dy. \qquad \ldots(44)$$

For the velocity component u we assume

$$u = U_0 f(\eta) \qquad \ldots(45)$$

in $b_c < y < b$, where $\eta = (y - b_c)/(b - b_c)$ and $f(\eta)$ is given by eqn. (32). Using (45) eqns. (41) and (44) respectively become

$$b_0^2 = \frac{20}{33} b_c^2 + \frac{32}{99} bb_c + \frac{7}{99} b^2 \qquad \ldots(46)$$

and

$$b_m^2 = \frac{10}{21} b_c^2 + \frac{8}{21} bb_c + \frac{1}{7} b^2. \qquad \ldots(47)$$

Now applying relations (45), (46) and (47) we get from eqn. (42)

$$\frac{d(b_c/b_0)}{d(x/b_0)} = \frac{530621}{5} k' I_1$$

$$\times \frac{G\left\{\left(\frac{23}{7} - \frac{I_2}{I_1}\right)\frac{b_c}{b_0} - G\right\}\left[\frac{1}{7\sqrt{21}}\left\{2079 - 280\,G\frac{b_c}{b_0} - 130\left(\frac{b_c}{b_0}\right)^2\right\}^{1/2} - \frac{b_c}{b_0}\right]^n}{\left(G - \frac{23}{7}\frac{b_c}{b_0}\right)^n\left[27027 + 6062\,G\frac{b_c}{b_0} - 12792\left(\frac{b_c}{b_0}\right)^2\right]} \qquad \ldots(48)$$

where

$$G = \left[\frac{99}{7} - \frac{164}{49}\left(\frac{b_c}{b_0}\right)^2\right]^{1/2}, \quad I_2 = \int_0^1 (-f')^{n+1}\, d\eta$$

and I_1 is already defined after eqn. (37).

Under the initial condition $b_c = b_0$ at $x = 0$ integration of eqn. (48) gives b_c as a function of x from which a radius of the jet b is to be determined.

Discussion

To study the established flow in two-dimensional jet the boundary condition that $U = U_0$ at $x = x_1$ is to be determined through experiment. There at every x the velocity distribution $\frac{u}{U}$ is plotted against $\frac{y}{b_0}$ and the area $F\left(= \int_0^\infty \frac{u}{U}\, d\left(\frac{y}{b_0}\right)\right)$ covered by the curve $\frac{u}{U} = f\left(\frac{y}{b_0}\right)$ and the axis $\frac{y}{b_0}$ is calculated. According to eqn. (9)

$$\frac{b_m}{b_0} = \int_0^\infty \frac{u}{U}\, d\left(\frac{y}{b_0}\right) = F$$

and so for different x the corresponding areas are substituted for $\frac{b_m}{b_0}$. Hence a relation between $\frac{b_m}{b_0}$ and x is to be established tentatively and comparing this relation with eqn. (15) along with eqn. (11) the value of x_1 is determined.

For axi-symmetric jet flow the same technique is applied and the value of x_1 is calculated.

Integrals I in eqns. (14), (15) and I_1, I_2 in eqns. (38), (39), (40) and (48) are solved over an IBM 1130 electronic computer by using Simpson's $\frac{1}{3}$ rule for different values of the power law index n. Some of the values so calculated are given in Table I.

TABLE I

n	I	I_1	I_2
0.8	1.27505	0.435386	1.43056
0.9	1.32177	0.454972	1.50612
1.0	1.37142	0.476190	1.58730
1.1	1.42411	0.499111	1.67443
1.2	1.47995	0.523819	1.76788

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REFERENCES

Hatta, K., and Nozaki, T. (1975). Two-dimensional and axi-symmetric jet flows with finite initial cross-sections. *Bull. JSME*, **18**, 349.

McCormick, John M., and Salvadori, Mario G. (1971). Numerical Methods in Fortran. Prentice-Hall of India Private Limited, New Delhi.

Schlichting, H. (1955). Boundary Layer Theory. McGraw-Hill Book Co., Inc., New York.

Skelland, A. H. P. (1967). Non-Newtonian Flow and Heat Transfer. John Wiley and Sons, Inc., New York.