# ON ABSOLUTE NÖRLUND SUMMABILITY OF LAGUERRE SERIES AT ORIGIN

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A new theorem on absolute Nörlund summability of Laguerre series at $x = 0$, has been proved under the bounded variation property of the generating function. Particular cases have also been deduced.

§1. Let $\Sigma a_n$ be an infinite series with $n$th partial sum $S_n$. The series $\Sigma a_n$ is said to be summable to $S$ by the Nörlund method $(N, p_n)$, if

$$t_n \equiv \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_n \to S \qquad ...(1.1)$$

as $n \to \infty$, where $P_n = \sum_{\nu=0}^{n} p_\nu$ and $P_n \neq 0$. The series is said to be absolutely summable-$(N, p_n)$ or summable - $| N, p_n |$ if the sequence $\{t_n\}$ is of bounded variation, i.e. if

$$\sum_n | t_{n+1} - t_n | < \infty. \qquad ...(1.2)$$

The Nörlund mean reduces to $(C, 1)$ mean when $p_\nu = 1$, for every $\nu$. If

$$p_n = \binom{n + \alpha - 1}{n}, \ \alpha > 0 \qquad ...(1.3)$$

the Nörlund mean $\{t_n\}$ reduces to $(C, \alpha)$-mean.

§2. The Fourier-Laguerre expansion corresponding to a function $f(x) \in L[0, \infty)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) \qquad ...(2.1)$$

where

$$\Gamma(\alpha + 1) \binom{n + \alpha}{n} a_n = \int_0^\infty e^{-x} . t^\alpha f(t) L_n^{(\alpha)}(t) \, dt \qquad ...(2.2)$$

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and $L_n^{(\alpha)}(x)$ denotes the Laguerre polynomials of order $\alpha > -1$, defined by the generating function

$$(1-\omega)^{-\alpha-1} \exp\left(-\frac{x\omega}{1-\omega}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, \omega^n. \qquad \text{...(2.3)}$$

We write

$$\phi(u) \equiv [f(u) - f(0)] \frac{e^{-u} u^{\alpha}}{\Gamma(\alpha+1)} \qquad \text{...(2.4)}$$

and

$$F(u) = [f(u) - f(0)] \cdot \frac{e^{-u} u^{(2\alpha-1)/4}}{\Gamma(\alpha+1)}\,. \qquad \text{...(2.5)}$$

§3. Recently Gupta (1970) has proved the following theorem on ordinary Cesàro summability of Laguerre series :

*Theorem A* — If

$$\int_0^t |\, \phi(u) \,|\, du = O(t^{1+\alpha}) \qquad \text{...(3.1)}$$

and

$$\int_1^{\infty} e^{t/2}\, t^{-k-1/3} \,|\, \phi(t) \,|\, dt < \infty \qquad \text{...(3.2)}$$

then the series (2.1) is $(C, k)$-summable at the point $x = 0$ with the sum $f(0)$ provided that $k > \alpha + \frac{1}{2}$ and $\alpha > -1$.

Very recently, Chaudhary (1974) proved the following result on non-local property of absolute Cesàro summability of Laguerre series.

*Theorem B* — The summability $|\, C, \alpha/2 + 3/4 \,|$, $(\alpha > -1)$ of the series (2.1) at $x = 0$ is not a local property in the sense that even if it is supposed that $f(x) = 0$ in some neighbourhood of $x = 0$ and $\infty$, the series need not be summable $|\, C, \alpha/2 + 3/4 \,|$ at $x = 0$.

Now it is clear that the series (2.1) is not necessarily summable $|\, C, \alpha/2 + 3/4 \,|$, even if $\phi(x) = 0$ at $x = 0$. Restricting $\phi(x)$ at $x = 0$ and imposing $BV$ property we prove the following theorem :

*Theorem* 1 — Let $\{p_n\}$ be a non-negative non-increasing sequence such that $P_n \to \infty$ and $\sum \frac{n^{(2\alpha-1)/4}}{P_n}$ is convergent. Then the series (2.1) is $|\, N, p_n \,|$-summable provided that for some small positive number $\epsilon$,

$$\phi(u) \in BV[0, \infty) \qquad \ldots(3.3)$$

$$\int_{n}^{\infty} e^{-t/2}\, t^{(6\alpha-7)/12}\, |f(t)|\, dt = O(n^{-1/2}) \qquad \ldots(3.4)$$

and $F(u)$ is bounded in $[0, \epsilon)$.

*Remarks* 1 : It is noticeable that for every function $f(x)$ (say) which is of bounded variation in $[0, \infty)$, the function $x^{-(2\alpha+1)/4} f(x)$ may not be bounded in the neighbourhood of $x = 0$. It is easily seen for $f(x) \equiv x^a \sin x^{-b}$. This function is of $BV$ in $[0, \epsilon)$ for $a > b > 0$. But obviously $x^{-(2\alpha+1)/4} f(x)$ is not necessarily bounded in $[0, \epsilon)$. In our theorem, therefore, the conditions that $\phi(u) \in BV[0, \infty)$ and $F(u) = u^{-(2\alpha+1)/4} \phi(u)$ bounded in $[0, \epsilon)$ are independent of each other.

*Remark* 2 : Theorem 1 is also a substitute of Theorem B in the sense that, for $F(u)$ bounded and $\phi(u) \in BV$, we obtain $|C, \delta|$ -summability of (2.1) at $x = 0$ for $\delta > (\alpha/2 + 3/4)$, $(\alpha > -1)$.

§4. We shall use the following orders of Laguerre polynomials :

If $\alpha$ be arbitrary and real and $c$ and $\omega$ are fixed positive constants, then

$$L_n^{(\alpha)}(x) = \begin{cases} x^{-(2\alpha+1)/4} \,.\, O(n^{(2\alpha-1)/4}), & \text{if } \frac{c}{n} \leqslant x \leqslant \omega, \\ O(n^{\alpha}) & \text{if } 0 \leqslant x \leqslant \frac{c}{n}, \end{cases} \qquad \ldots(4.1)$$

as $\quad n \to \infty$.

For $\alpha$, $\lambda$ arbitrary and real, $a > 0$, $0 < \eta < 4$, we have

$$\max e^{-x/2} x^{\lambda} \,.\, |L_n^{(\alpha)}(x)| \sim n^{Q},\ n \to \infty, \qquad \ldots(4.2)$$

where

$$Q = \begin{cases} \max\left(\lambda - \tfrac{1}{2}, \frac{\alpha}{2} - \tfrac{1}{4}\right); & a \leqslant x \leqslant (4-\eta)n \\ \max\left(\lambda - \tfrac{1}{3}, \frac{\alpha}{2} - \tfrac{1}{4}\right), & x \geqslant a. \end{cases} \qquad \ldots(4.3)$$

It is well known that for arbitrary and real $\alpha$,

$$\frac{d}{dx}\{L_n^{(\alpha)}(x)\} = -L_{n-1}^{(\alpha+1)}(x). \qquad \ldots(4.4)$$

Reference for all these results may be made to Szegö (1967). Besides these, we need the following lemma to complete the proof of our theorem.

*Lemma* 1 (Bhatt 1962) — Let $\Sigma\, u_n$ be an infinite series with $n$th partial sum $S_n$. Let $\{p_n\}$ be a positive non-increasing sequence such that $P_n \to \infty$. Then, if

$$\sum_n \frac{|S_n|}{P_n} < \infty \qquad \ldots(4.5)$$

the series $\Sigma\, u_n$ is summable $|N, p_n|$.

PROOF OF THE THEOREM: On account of the relation

$$\sum_{\nu=0}^{n} L_\nu^{(\alpha)}(x) = L_n^{(\alpha+1)}(x),$$

the $n$th partial sum of the series (2.1) is given by

$$S_n = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-u} u^\alpha f(u)\, L_n^{(\alpha+1)}(u)\, du.$$

Again, due to the orthogonality of Laguerre polynomials, we have

$$S_n - f(0) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty \phi(u)\; L_n^{(\alpha+1)}(u)\, du$$

$$= \frac{1}{\Gamma(\alpha+1)} \left\{ \int_0^{c/n} + \int_{c/n}^{\varepsilon} + \int_{\varepsilon}^{n} + \int_{n}^{\infty} \right\},$$

($c$ being small positive number)

$$= S_{n\cdot 1} + S_{n\cdot 2} + S_{n\cdot 3} + S_{n\cdot 4}, \text{ say.}$$

Now

$$S_{n\cdot 1} = O(n^{\alpha+1}) \int_0^{c/n} |\phi(u)|\, du$$

$$= O(n^{\alpha+1}) \int_0^{c/n} u^{(2\alpha+1)/4} \cdot |F(u)|\, du$$

$$= O(n^{\alpha+1}) \cdot (n^{-(2\alpha+5)/4}) \text{ (since } |F(u)| = O(1)),$$

$$= O(n^{(2\alpha-1)/4}). \qquad \ldots(4.6)$$

$$S_{n\cdot 2} = \frac{1}{\Gamma(\alpha+1)} \int_{1/n}^{\varepsilon} \phi(u) \cdot \frac{d}{du} \{-L_{n-1}^{(\alpha)}(u)\}\, du$$

(*equation continued on p.* 771)

$$= \frac{-1}{\Gamma(\alpha+1)} [\phi(u) \,.\, L_{n-1}^{(\alpha)}(u)]_{c/n}^{\varepsilon}$$

$$+ \frac{1}{\Gamma(\alpha+1)} \int_{c/n}^{\varepsilon} L_{n-1}^{(\alpha)}(u)\, d\phi(u)$$

$$= O\{F(u) \,.\, n^{(2\alpha-1)/4}\} + \frac{1}{\Gamma(\alpha+1)} \int_{c/n}^{\varepsilon} L_{n-1}^{(\alpha)}(u)\, d\phi(u).$$

But

$$\left| \int_{1/n}^{\varepsilon} L_n^{(\alpha)}(u) \,.\, d\phi(u) \right| \leqslant \left| \int_{1/n}^{\varepsilon} d\{L_n^{(\alpha)}(u)\, \phi(u)\} \right|$$

$$+ \left| \int_{1/n}^{\varepsilon} \phi(u)\, dL_n^{(\alpha)}(u) \right|$$

$$= \left| [L_n^{(\alpha)}(u)\, \phi(u)]_{1/n}^{\varepsilon} + O(n^{(2\alpha-3)/4}). \right.$$

$$\times \int_{c/n}^{\varepsilon} |\phi(u)|\, u^{-(2\alpha-1)/4}\, du$$

$$= O(n^{(2\alpha-1)/4}) + O(n^{(2\alpha-3)/4}) \int_{1/n}^{\varepsilon} |F(u)|\, u^{1/2}\, du$$

$$= O(n^{(2\alpha-1)/4}).$$

Therefore, we have

$$S_{n \cdot 2} = O(n^{(2\alpha-1)/4}). \qquad \ldots(4.7)$$

Again

$$S_{n \cdot 3} = \frac{1}{\Gamma(\alpha+1)} \int_{\varepsilon}^{n} \phi(u)\, L_n^{(\alpha+1)}(u)\, du$$

$$= \frac{-1}{\Gamma(\alpha+1)} \int_{\varepsilon}^{n} \phi(u) \frac{d}{du} \{L_{n-1}^{(\alpha)}(u)\}\, du$$

$$= \frac{-1}{\Gamma(\alpha+1)} [\phi(u)\; L_{n-1}^{(\alpha)}(u)]_{\varepsilon}^{n} + \frac{1}{\Gamma(\alpha+1)} \int_{\varepsilon}^{n} L_{n-1}^{(\alpha)}(u)\, d\phi(u)$$

$$= O(n^{(2\alpha-1)/4}) + O(n^{(2\alpha-1)/4}) \int_{\varepsilon}^{n} |d\phi(u)|$$

$$= O(n^{(2\alpha-1)/4}). \qquad \ldots(4.8)$$

$$S_{n,4} = \int_n^{\infty} \phi(u) \,.\, L_n^{(\alpha+1)}(u)\, du$$

$$= O(n^{(2\alpha+1)/4}) \,.\, \int_n^{\infty} e^{-(u/2)}\, u^{(6\alpha-7)/12} \,.\, | f(u) |\, du$$

$$= O(n^{(2\alpha-1)/4}). \qquad \ldots(4.9)$$

Collecting the results (4.6), (4.7), (4.8) and (4.9), we have

$$\sum_n \frac{| S_n - f(0) |}{P_n} \cong \sum_n O\left\{\frac{n^{(2\alpha-1)/4}}{P_n}\right\}$$

consequently the result follows in view of Lemma 1.

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