PROOF OF THE LITTLEWOOD CONJECTURE FOR INFINITELY MANY PAIRS

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The "Littlewood Conjecture" is the following : Given two irrational numbers u_1, u_2, there exists an infinite sequence of triples of integers p, q, r which make $| q | \; | u_1 q - p | \; | u_2 q - r |$ arbitrarily small. Call u_1, u_2 a P.V.J.P. pair if the Jacobi-Perron algorithm is purely periodic, positive, convergent and the characteristic equation has one real root which is a P.V. number. The author proves the following :

Theorem — If u_1, u_2 is P.V.J.P., then there exists an infinite sequence of triples p, q, r satisfying $| q | \; | u_1 q - p | \; | u_2 q - r | << \ln^{-1}(q)$. There are infiniteley many P.V.J.P. pairs. An example is given by the pair x, $x^2 - x$ where x is the real root of $x^3 - x^2 - x - 1 = 0$. Here the algorithm is purely periodic of period length one.

In this paper we will use the notation of Perron (1907). Let the pair u_1, u_2 of real numbers have a (purely) periodic Jacobi-Perron Algorithm of length k. Let positive algorithm be an algorithm so that the period $(a(j), b(j))$ $j = 0, \ldots, k - 1$; has $a(j) \geqslant 0$, $b(j) \geqslant 1$ and $a(j)$, $b(j)$ integers. From the period one defines $a(j)$, $b(j)$ for all integers j by $a(j + rk) = a(j)$ and $b(j + rk) = b(j)$; $r = 1, \ldots, \infty$; $j = 0, \ldots, k - 1$. Three sequences of integers are defined as follows:

$$A_i(j+3) = A_i(j) + a(j)\, A_i(j+1) + b(j)\, A_i(j+2);\ i = 0, 1, 2;$$
$$j = 0, \ldots, \infty \text{ and where } A_i(i) = 1;\ A_i(j) = 0,$$
$$0 \leqslant i \neq j \leqslant 2. \qquad \ldots(1)$$

As a consequence of Theorem 2 of Perron (1907) a positive periodic Jacobi-Perron algorithm is convergent. For convergent periodic algorithms the characteristic equation is formed from the matrix :

$$\begin{vmatrix} A_0(k) & A_0(k+1) & A_0(k+2) \\ A_1(k) & A_1(k+1) & A_1(k+2) \\ A_2(k) & A_2(k+1) & A_2(k+2) \end{vmatrix} \qquad \ldots(2)$$

The determinant of the matrix is equal to one (Perron 1907, p. 6). Let the characteristic equation be $x^3 - bx^2 - ax - 1 = 0$, $b \geqslant 1$. By Theorem 9 of Perron (1907):

(a) The characteristic equation has at least one positive simple real root σ.

(b) If the algorithm is positive then the simple root σ has absolute value larger than the other two roots.

(c) $\lim \dfrac{A_2(j)}{A_0(j)} = \dfrac{x_2}{x_0} = u_2$

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$\lim \dfrac{A_0(j+k)}{A_0(j)} = \sigma$, where $x_0, x_1, x_2 > 0$ and the ratio $x_0 : x_1 : x_2$ is determined by the following system of homogeneous linear equations :

$\sigma x_i = A_i(k)\, x_0 + A_i(k+1)\, x_1 + A_i(k+2)\, x_2;\ i = 0, 1, 2.$

If we define $H_i(v, \lambda) = A_i(kv + \lambda) - u_i A_0(kv + \lambda)$.

Theorem 17 of Perron states :

$$| H_i(v, \lambda) | \leqslant (A_0(kv + \lambda))^{-1/2},\ 0 \leqslant \lambda < k \qquad \text{...(3)}$$

holds if and only if the characteristic equation has only one real root σ and $\sigma > 1$. If we call α, β the two complex roots then $1 > | \alpha | = | \beta | = 1/\sqrt{\sigma}$. Therefore if (3) is satisfied then σ is a P.V. number. Call u_1, u_2 a P.V.J.P. pair if the Jacobi-Perron algorithm is positive periodic and the characteristic equation has only one simple real root $\sigma > 1$.

We shall consider only P.V.J.P. pair in what follows. Let $\triangle$ be the square root of the discriminant of the characteristic equation. Consequently $\triangle$ is purely imaginary and

$$\triangle = \begin{vmatrix} 1 & \sigma & \sigma^2 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix} = (\alpha - \sigma)(\beta - \sigma)(\beta - \alpha). \qquad \text{...(4)}$$

Perron (1907, p. 42) has the following formula for $A_i(vk + \lambda)$.

$$A_i(vk + \lambda) = - \frac{\begin{vmatrix} 0 & A_i(\lambda) & A_i(k+\lambda) & A_i(2k+\lambda) \\ \sigma^v & 1 & \sigma & \sigma^2 \\ \alpha^v & 1 & \alpha & \alpha^2 \\ \beta^v & 1 & \beta & \beta^2 \end{vmatrix}}{\triangle};\ i = 0, 1, 2.$$

We then have $\triangle H_i(v, \lambda)$ equal to

$$\begin{vmatrix} 0 & A_0(\lambda) & A_0(k+\lambda) & A_0(2k+\lambda) \\ \sigma^v & 1 & \sigma & \sigma^2 \\ \alpha^v & 1 & \alpha & \alpha^2 \\ \beta^v & 1 & \beta & \beta^2 \end{vmatrix} - \begin{vmatrix} 0 & A_i(\lambda) & A_i(k+\lambda) & A_i(2k+\lambda) \\ \sigma^v & 1 & \sigma & \sigma^2 \\ \alpha^v & 1 & \alpha & \alpha^2 \\ \beta^v & 1 & \beta & \beta^2 \end{vmatrix} \quad \text{...(5)}$$

If the two determinants of $H_i(v, \lambda)$ are expanded along their first column, we will have, $\triangle H_i(v, \lambda) = \sigma^v T_0 - \alpha^v T_1 + \beta^v T_2$. Now $T_0 = 0$ for each i, λ by Perron (1907, p. 63),

$$\triangle H_i(v, \lambda) = -\alpha^v T_1 + \beta^v T_2. \quad \text{...(6)}$$

Lemma 1 — T_1 is the complex conjugate of T_2.

PROOF : T_2 is obtained by replacing β for α in T_1.

Let $\beta = |\beta| \exp(i\theta)$ and $T_1 = |T_1| \exp(iw)$. Since $\triangle$ is purely imaginary and $H_i(v, \lambda)$ is real, we have

$$H_i(v, \lambda) = i \triangle^{-1} \operatorname{Im}(-\alpha^v T_1 + \beta^v T_2). \quad \text{...(7)}$$

But $\quad \operatorname{Im}(-\alpha^v T_1 + \beta^v T_2) = |\beta|^v \, |T_1| \operatorname{Im}(\exp(i\theta v - iw) - \exp(-i\theta v + iw))$

$$= 2 |\beta|^v |T_1| \sin(\theta v - w).$$

Hence,

$$H_i(v, \lambda) = 2 |\triangle|^{-1} |\beta|^v |T_1| \sin(\theta v - w) = C |\beta|^v \sin(\theta v - w) \quad \text{...(8)}$$

where C is a constant independent of v.

Consider the linear form $(\theta/\pi)x - (w/\pi)$. By Minkowski's Theorem (Cassels 1957, p. 48), there are infinitely many pairs of integers such that,

$$|v| \left| \frac{v\theta}{\pi} - \frac{w}{\pi} - n \right| < \frac{1}{4}, \text{ for } n, v \text{ integers.} \quad \text{...(9)}$$

So for infinitely many pairs of integers v, n,

$$|v| \, |v\theta - w - \pi n| < \frac{\pi}{4} \text{ and therefore for these } n, v;$$

$$|\sin(\theta v - w)| = \sin |\theta v - w - \pi n| < |\theta v - w - \pi n| \quad \text{...(10)}$$

since $\quad \sin(x) < x$ for $\theta < x < \dfrac{\pi}{2}$ ·

Lemma 2 — If c denotes a constant independent of v, then

(1) $\sigma^v \gg cA_0(vk + \lambda), \lambda < k$

(2) $v \gg c \ln (A_0(vk + \lambda)), \lambda < k.$

PROOF: Let $\lambda/k = t$ for λ and k fixed, then from (c) of page 1,

$$A_0(j + \lambda) \ll \sigma^t A_0(j),\ A_2(j + \lambda) \ll u_2\sigma^t A_0(j) \text{ and } A_1(j + \lambda) \ll u_1\sigma^t A_0(j),\ j \text{ fixed. Hence}$$

$$A_0(vk) + A_1(vk + 1) + A_2(vk + 2) \gg \sigma^{-t}A_0(vk + \lambda) + u_1\sigma^{(1/k)-t} A_0(vk + \lambda) + u_2\sigma^{(2/k)-t} A_0(vk + \lambda) \gg cA_0(vk + \lambda).$$

Since (Perron 1907, p. 32)

$$\sigma^v + \alpha^v + \beta^v = A_0(vk) + A_1(vk + 1) + A_2(vk + 2)$$

and $$1 > |\alpha| = |\beta|,\ \sigma > 1,$$

we have $\sigma^v \gg cA_0(vk + \lambda)$. Taking logarithms, (2) follows.

Theorem 1 — There exist infinitely many triples of integers p, q, so that

$$\sqrt{q} \ln (q) \mid p - u_i q \mid \ll 1. \quad \ldots(11)$$

PROOF: Choose a sequence of pairs of integers v, n satisfying (9) then,

$$\sqrt{A_0(kv + \lambda)} \ln (A_0(kv + \lambda)) \mid A_i(kv + \lambda) - u_i A_0(kv + \lambda) \mid$$
$$= c\sqrt{A_0(kv + \lambda)} \ln (A_0(kv + \lambda)) \mid \beta \mid^v \mid \sin (\theta v - w) \mid, \text{ by (8)}$$
$$= c\sigma^{v/2} \mid v \mid \sigma^{-v/2} \sin (\theta v - w), \text{ by the previous lemma}$$
$$\ll c \mid v \mid \ \mid \sin (\theta v - w) \mid < c, \text{ by (9) and (10).}$$

Put $q = A_0(kv + \lambda)$ and $p = A_i(kv + \lambda)$ and the theorem follows.

Theorem 2 — There exist infinitely many triples of integers p, q, r so that

$$q \ln (q) \mid p - u_2 q \mid \ \mid r - u_1 q \mid \ll 1. \quad \ldots(12)$$

PROOF: Theorem 1 and statement (3).

Corollary — If u_1 u_2 is a P.V.J.P. pair, then the "Littlewood Conjecture" is true for u_1, u_2.

Theorem 3 — There exist infinitely many P.V.J.P. pairs. This follows from the following lemma.

Lemma 3 — If $1 \leqslant a^2 < 4b$, a, b being positive integers, and σ is a root of

$$x^3 - bx^2 - ax - 1 = 0,$$

then the pair σ, $\sigma^2 - b\sigma$ is P.V.J.P.

PROOF : From Theorem 7 of Perron we need only to show $\sigma > 1$ and the other two roots of $x^3 - bx^2 - ax - 1 = 0$ are complex. If we write

$$x^3 - bx^2 - ax - 1 = (x - \sigma)(x^2 + (\sigma - b)x + 1/\sigma),$$

then σ^4 times the discriminant D of $x^2 + (\sigma - b)x + 1/\sigma$ will be

$$\sigma^4 D = \sigma^4\left((\sigma - b)^2 - \frac{4}{\sigma}\right) = (\sigma^3 - b\sigma^2)^2 - 4\sigma^3$$
$$= a^2\sigma^2 + 2a\sigma + 1 - 4b\sigma^2 - 4a\sigma - 4$$
$$= (a^2 - 4b)\sigma^2 - 2a\sigma - 3 < 0,$$

since $a^2 < 4b$.

Example : If one chooses $x \simeq 1.83928675521$, the positive real root of

$$x^3 - x^2 - x - 1 = 0 \text{ and } y = x^2 - x = 1.54368901268 \text{ (Lemma 3)},$$

then (i) x, y is P.V.J.P. (ii) x, y has a periodic Jacobi-Perron algorithm of period length 'one' and (iii) by Theorem 2 there exist infinitely many triples of integers p, r, q, so that

$$q \ln(q) \mid p - xq \mid \mid r - yq \mid \leqslant 1$$

Remark : The author has verified (for $q \leqslant 2^{35} - 1$) the cyclic nature of $H_i(\nu, \lambda)$ for some pairs obtained from Lemma 3 using a 360/67 systems computer.

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