ON GENERALIZED QUASI-CONVEX SEQUENCE AND ITS APPLICATIONS

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(*Received* 12 *January* 1976)

The purpose of this note is to define a generalized quasi-convex sequence and study some of its applications in the theory of summability and Fourier analysis.

§1. Given a sequence $\{a_n\}$ we write $\triangle a_n = a_n - a_{n+1}$, $\triangle^m a_n = \triangle(\triangle^{m-1} a_n)$ with $\triangle^0 a_n = a_n$, where m is a positive integer. The sequence $\{a_n\}$ is said to be convex if $\triangle^2 a_n \geqslant 0$. It is well known that if $\{a_n\}$ is bounded and convex, then

$$a_n \downarrow,\ n\triangle a_n \to 0,\ n \to \infty \text{ and } \sum_{1}^{\infty} (n+1) \triangle^2 a_n < \infty.$$

For other interesting results reference may be made to Chow (1941), Pati (1954, 1962), Prasad and Bhatt (1957), Bhatt (1962) and Mazhar (1966a, b).

A sequence $\{a_n\}$ is said to be quasi-convex if

$$\sum_{1}^{\infty} (n+1) \mid \triangle^2 a_n \mid < \infty. \qquad \text{...(1.1)}$$

It is clear from the above result that every bounded convex sequence is quasi-convex. However, the converse need not be true. Contrary to what we have for convex sequences, a null quasi-convex sequence $\{a_n\}$ need not be monotonic decreasing. It is, however, of bounded variation and it satisfies the condition

$$n\triangle a_n \to 0,\ n \to \infty.$$

The concept of quasi-convex sequence was recently generalized by Telyakovskii (1973). According to him a sequence $\{a_n\}$ is said to belong to class S if

(i) $a_n \to 0,\ n \to \infty$,

(ii) there exists a sequence of numbers $\{A_k\}$ such that $A_k \downarrow 0$ and $\sum_{1}^{\infty} A_k < \infty$,

(iii) $\mid \triangle a_k \mid \leqslant A_k$ for all k.

Taking $A_k = \sum_{m=k}^{\infty} \mid \triangle^2 a_m \mid$ it follows that a null quasi-convex sequence $\{a_n\}$ belongs to the class S. The converse is obviously not true. In view of the conditions (ii) and (iii), it follows that every sequence $\{a_n\}$ of class S is of bounded variation and that $n \triangle a_n \to 0$, as $n \to \infty$.

In this note we propose to obtain a more general class by introducing the concept of δ-quasi-monotonicity. A sequence $\{b_n\}$ of positive numbers is said to be quasi-monotone if $\triangle b_n \geqslant -\alpha b_n/n$ for some positive α. It is obvious that every null monotonic decreasing sequence is quasi-monotone. The sequence $\{b_n\}$ is said to be δ-quasi-monotone if $b_n \to 0$, $b_n > 0$ ultimately and $\triangle b_n \geqslant -\delta_n$, where $\{\delta_n\}$ is a sequence of positive numbers. Clearly a null quasi-monotone sequence is δ-quasi-monotone with $\delta_n = \alpha b_n/n$.

A sequence $\{a_n\}$ will be said to belong to class $S(\delta)$ if

(a) $a_n \to 0,\ n \to \infty$,

(b) there exists a sequence of numbers $\{A_n\}$ such that it is δ-quasi-monotone and $\sum_1^\infty A_n$ is convergent,

(c) $|\triangle a_n| \leqslant |A_n|$ for all n.

It is obvious that $a_n \in S \Rightarrow a_n \in S(\delta)$.

The concepts of convex and quasi-convex sequences have been applied to various types of problems in different branches of Mathematics, such as Theory of Summability, Fourier Analysis etc. In Section A of this note, we shall study an application of the generalized quasi-convex sequence to a problem in the theory of absolute summability factors, while in Section B, we discuss its application to a well-known problem in Fourier analysis.

Section A

§2. Let $\sum_1^\infty a_n$ be a given infinite series with s_n as its nth partial sum. We denote by t_n the nth $(C, 1)$ mean of the sequence $\{na_n\}$. The series $\sum_1^\infty a_n$ is said to be summable

$|C, 1|_k$, $k \geqslant 1$, if $\sum_1^\infty \frac{|t_n|^k}{n} < \infty$ (Flett 1957).

Generalizing a theorem of Pati (1962), Mazhar (1966 b) (see also Mishra 1965) proved the following theorem :

Theorem A — If

$$\triangle^2 \lambda_n \geqslant 0 \text{ and } \sum_1^\infty \frac{\lambda_n}{n} < \infty \qquad \ldots(2.1)$$

and

$$\sum_{1}^{m} \frac{|s_r|^k}{r} = O(\log m), \; m \to \infty, \; k \geqslant 1 \qquad \ldots(2.2)$$

then the series $\sum_{1}^{\infty} a_n \lambda_n$ is summable $|C, 1|_k$.

Later on, Mazhar (1972) proved a more general theorem in which he replaced the conditions (2.1) and (2.2) by the following :

$$\lambda_n \to 0, \; \sum_{1}^{\infty} n \log n \, |\Delta^2 \lambda_n| < \infty \qquad \ldots(2.3)$$

$$\sum_{1}^{m} \frac{|t_n|^k}{n} = O(\log m), \; m \to \infty. \qquad \ldots(2.4)$$

It is known (Pati 1962, Mazhar 1966) that (2.1) $\Rightarrow$ (2.3) and it is easy to show that (2.2) $\Rightarrow$ (2.4). In this section we propose to show how condition (2.3) can be further relaxed by using the concept of generalized quasi-convex sequences. Our theorem is as follows.

Theorem 1 — Let $\lambda_n \to 0$, $n \to \infty$. Suppose that there exists a sequence of numbers $\{A_k\}$ such that it is δ-quasi-monotone with $\sum_{1}^{\infty} n \log n \, \delta_n < \infty$, $\sum_{1}^{\infty} A_k \log k$ is convergent and $|\Delta \lambda_k| \leqslant |A_k|$ for all k. If (2.4) holds, then $\sum_{1}^{\infty} a_n \lambda_n$ is summable $|C, 1|_k$.

To show that condition (2.3) implies the condition of our theorem we can take, for example, $A_k = \sum_{n=k}^{\infty} |\Delta^2 \lambda_n|$.

§3. We need the following lemma for the proof of Theorem 1.

Lemma 1 — If $\{b_n\}$ is δ-quasi-monotone with $\sum n \log n \, \delta_n < \infty$ and $\sum b_n \log n$ is convergent, then

$$m b_m \log m \to 0, \; m \to \infty \qquad \ldots(3.1)$$

$$\sum n \log n \, |\Delta b_n| < \infty. \qquad \ldots(3.2)$$

The proof of this lemma is similar to that of Theorems 1 and 2 of Boas (1965, case $\gamma = 1$) and hence omitted.

§4. *Proof of Theorem* 1 — By partial summation

$$T_n = \frac{1}{n+1} \sum_{1}^{n} v a_v \lambda_v = \frac{1}{n+1} \sum_{v=1}^{n-1} \triangle \lambda_v \sum_{r=0}^{v} r a_r + \frac{\lambda_n}{n+1} \sum_{r=0}^{n} r a_r$$

so that

$$\sum_{1}^{m} \frac{|T_n|^k}{n} = O\Big[\sum_{1}^{m} n^{-k-1} \Big(\sum_{v=1}^{n-1} |\triangle \lambda_v| (v+1) |t_v| \Big)^k\Big]$$

$$+ O\Big[\sum_{1}^{m} \frac{|\lambda_n|}{n} |t_n|^k\Big]$$

$$= O[\sum_{1}^{m} n^{-k-1} (\sum_{v=1}^{n-1} v |A_v| \; |t_v|^k) (\sum_{v=1}^{n-1} v |A_v|)^{k-1}]$$

$$+ O\Big[\sum_{1}^{m} \frac{|t_n|^k}{n} \sum_{v=n}^{\infty} |\triangle \lambda_v|\Big]$$

$$= O\Big[\sum_{1}^{m} n^{-2} \sum_{v=1}^{n-1} v |A_v| \; |t_v|^k\Big] + O\Big[\sum_{n=1}^{m} \frac{|t_n|^k}{n} \sum_{v=n}^{\infty} |A_v|\Big]$$

$$= O\Big[\sum_{v=1}^{m} v |A_v| \frac{|t_v|^k}{v}\Big] + O\Big[\sum_{v=1}^{\infty} |A_v| \sum_{n=1}^{v} \frac{|t_n|^k}{n}\Big]$$

$$= O\Big[\sum_{v=1}^{m-1} \triangle (v |A_v|) \sum_{r=1}^{v} \frac{|t_r|^k}{r} + m |A_m| \sum_{r=1}^{m} \frac{|t_r|^k}{r}\Big]$$

$$+ O[\sum_{v=1}^{\infty} |A_v| \log v]$$

$$= O[\sum_{v=1}^{m-1} (|A_v| + v |\triangle A_v|) \log v] + O(m |A_m| \log m) + O(1)$$

$$= O(1), \quad m \to \infty$$

by virtue of the hypothesis and Lemma 1.

Remark : Following the analysis of this note we can obtain a more general result in which $\log n$ is replaced by a positive sequence $\{\mu_n\}$ such that it is non-decreasing and

$$\mu_{n+1} - \mu_n = O\Big(\frac{\mu_n}{n}\Big).$$

SECTION B

§5. Let us consider the trigonometric cosine series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx. \qquad \ldots(5.1)$$

If $a_n \downarrow 0$, then this series converges to a function $f(x)$ for all x except possibly at $x = 0$. It is well known (Zygmund 1959) that the condition of monotonicity alone does not ensure the L-integrability of $f(x)$ and consequently (5.1) cannot be a Fourier series. Young (1913) proved that if $\{a_n\}$ is a null convex sequence, then the above series is a Fourier series of a non-negative function. Later on, Kolmogorov (1923) observed that for the series (5.1) to be a Fourier series it is enough to assume that $\{a_n\}$ is a null quasi-convex sequence. These results were subsequently generalized by Telyakovskii (1964, 1967) and others. The conditions imposed on $\{a_n\}$ were, however, quite involved. Recently Telyakovskii (1973) proved another theorem which is as follows :

Theorem B — Let $a_n \in S$, then (5.1) is a Fourier series and the following estimate is valid :

$$\int_0^{\pi} \left| \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nx \right| dx \leqslant C \sum_{0}^{\infty} A_k,$$

where C is an absolute constant.

It was shown by him that Theorem B can be deduced from his more general results obtained earlier. He has also observed that his result is equivalent to a theorem of Sidon (1939). However, his result (Theorem B) is interesting in the sense that conditions on $\{a_n\}$ are simple and can be verified easily.

In this section we propose to obtain a generalization of the above theorem by introducing the notion of δ-quasi-convex sequences.

We prove the following theorem.

Theorem 2 — Let $a_k \to 0$ and $\{A_k\}$ be a δ-quasi-monotone sequence with $\sum_1^{\infty} n\delta_n < \infty$. Suppose $\sum_1^{\infty} A_k$ is convergent and $| \Delta a_k | \leqslant | A_k |$ for all k. Then $a_0/2 + \sum_1^{\infty} a_n \cos nx$ is a Fourier series and

$$\int_0^{\pi} \left| \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nx \right| dx \leqslant C \sum_{0}^{\infty} | A_k | .$$

The following lemmas are pertinent for the proof of our theorem.

Lemma 2 (Boas 1965) — If $\{a_n\}$ is δ-quasi-monotone with $\Sigma\ n^\gamma \delta_n < \infty,\ \gamma \neq 0$, then the convergence of $\Sigma\ n^{\gamma-1}\ a_n$ implies that $n^\gamma a_n \to 0,\ n \to \infty$.

Lemma 3 (Telyakovskii 1973) — If the sequence of numbers $\{\alpha_i\}$ satisfies the condition $|\ \alpha_i\ | \leqslant 1$, then

$$\int_0^\pi \left| \sum_{i=0}^{k} \alpha_i D_i(x) \right| dx \leqslant C(k+1),$$

where $D_i(x) = \frac{1}{2} + \cos x + \cos 2x + \ldots \cos ix$.

Proof of Theorem 2 — Since $\sum_0^\infty A_k$ is convergent and $A_k > 0$ ultimately, it follows that $\sum_0^\infty |\ A_k\ | < \infty$. From the hypothesis, $\Sigma\ |\ \triangle\ a_k\ | < \infty$. Therefore, by the well-known result, $a_0/2 + \sum_1^\infty a_n \cos nx$ is convergent for all x except possibly at $x = 0$. Hence in order to prove that it is a Fourier series, it is sufficient to establish Lebesgue integrability of its sum-function.

Now

$$\frac{a_0}{2} + \sum_1^\infty a_n \cos nx = \sum_0^\infty \triangle\ a_n D_n(x)$$

$$= \sum_0^\infty A_n \frac{\triangle\ a_n}{A_n} D_n(x)$$

$$= \sum_0^\infty (A_n - A_{n+1}) \sum_0^n \frac{\triangle\ a_m}{A_m} D_m(x),$$

by virtue of Lemma 2 ($\gamma = 1$). Hence in view of Lemma 3

$$\int_0^\pi \left| \frac{a_0}{2} + \sum_1^\infty a_n \cos nx \right| dx$$

$$\leqslant \sum_0^\infty |\ A_n - A_{n+1}\ | \int_0^\pi \left| \sum_{m=0}^{n} \frac{\triangle\ a_m D_m(x)}{A_m} \right| dx$$

$$\leqslant C \sum_0^\infty (n+1)\ |\ A_n - A_{n+1}\ | = C \sum_0^\infty (n+1)\ |\ A_n - A_{n+1} + \delta_n - \delta_n\ |$$

$$\leqslant C \sum_0^\infty (A_n - A_{n+1} + \delta_n)(n+1) + C \sum_0^\infty \delta_n(n+1)$$

$$= C \sum_0^\infty (A_n - A_{n+1})(n+1) + 2 \sum_0^\infty (n+1)\delta_n$$

$$\leqslant C \sum_0^\infty |A_n|$$

since $\sum_0^\infty n\delta_n < \infty$.

This completes the proof of Theorem 2.

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