STABILITY CRITERIA FOR TIME-DEPENDENT FLOWS

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By analysing the linearized differential equations for the small perturbations, some of the instability criteria of the steady, inviscid theory are generalized to different classes of unsteady flows. It is shown that for time-dependent parallel flow profiles whose functional dependence is separable, a necessary condition for instability is that the vorticity of the basic flow must somewhere have a maximum in the flow domain throughout the time history of flow. Similar generalizations of instability criteria of Rayleigh (1892) and of Batchelor and Gill (1962) for axisymmetric jets are proved. Finally, Rayleigh's point-of-inflection criterion is extended to time-dependent flow profiles taking into consideration disturbances which vary spatially in the streamwise direction.

1. INTRODUCTION

The question as to what might be the effects in the stability analysis of laminar flows if the flow under investigation is unsteady has attracted attention recently. The importance of this question arises partly due to the fact that after a basic steady flow becomes unstable, a time-dependent flow subsequently results, an investigation of whose stability is equally important in understanding the process of transition to turbulence.

Conrad and Criminale (1965) showed that in inviscid fluids, for time-dependent flow profiles whose functional dependence is separable, a necessary condition for instability is that the velocity profile of the basic flow should have a point of inflection in the domain of flow. This was a generalization of a necessary condition for instability of two-dimensional steady flows, the so-called point-of-inflection criterion, given by Rayleigh (1880).

In the following analysis, we obtain some more extensions to the time-dependent case, of other instability criteria for steady flows.

2. EXTENSION OF THE FJϕRTOFT-HϕILAND CONDITION

Fjϕrtoft (1950) and Hϕiland (1953) showed that in parallel flow of inviscid fluids, only for velocity profiles the modulus of whose vorticity has a maximum in the domain of flow can there be a disturbance which grows with time. In extending this condition to the time-dependent case we start with the Rayleigh stability equation for infinitesimal disturbances to the flow of an inviscid, incompressible fluid of constant density moving with time-dependent velocity ($U(z, t)$, 0, 0) [see for example, Conrad and Criminale (1965)] :

$$\left(\frac{\partial}{\partial t} + ikU\right)(f'' - K^2 f) - ik\, U'' f = 0 \qquad \text{...(1)}$$

where $f(z, t)$ serves to determine the perturbation velocity component w perpendicular to the flow direction x and is given by

$$w = f(z, t)\, e^{ikx} \qquad \text{...(2)}$$

k being a constant and a prime denoting differentiation with respect to z measured vertically upwards.

If we suppose that the fluid is confined between two rigid planes at $z = 0$ and $z = d$ (say), then we must require that the solutions of eqn. (1) must satisfy the boundary conditions

$$f = 0 \text{ at } z = 0 \text{ and } z = d. \qquad \text{...(3)}$$

Suppose that the base flow $U(z, t)$ is separable. Hence we can assume that such is also the case for the perturbation velocity components and let

$$f(z, t) = \sum_{n=1}^{\infty} g_n(z)\, j_n(t). \qquad \text{...(4)}$$

Substituting this relation in eqn. (1) we get equations of the form

$$\left(\frac{dj_n}{d} + ikUj_n\right)(g_n'' - k^2 g_n) - ikU'' j_n g_n = 0.$$

Assuming j_n to be real and g_n to be complex, thereby losing no generality, we can write

$$g_n'' - k^2 g_n = \frac{ikU'' j_n g_n}{\dfrac{dj_n}{dt} + ikUj_n}. \qquad \text{...(5)}$$

Multiplying this equation by g_n^*, the complex conjugate of g_n, integrating from 0 to d, using the boundary conditions (3) and clearing the denominator of the right-hand side of complex numbers, we have

$$-\int_0^d (|g_n'|^2 + k^2 |g_n|^2)\, dz$$

$$= \int_0^d ikU'' |g_n|^2 j_n \frac{\dfrac{dj_n}{dt} - ikUj_n}{\left(\dfrac{dj_n}{dt}\right)^2 + (kUj_n)^2}\, dz. \qquad \text{...(6)}$$

The imaginary part of this equation gives

$$ikj_n \frac{dj_n}{dt} \int_0^d U'' \frac{|g_n|^2 \, dz}{\left(\frac{dj_n}{dt}\right)^2 + (kUj_n)^2} = 0$$

which shows that either $j_n = 0$ (there is no disturbance) or $dj_n/dt = 0$ (the disturbance cannot grow in time), or

$$\int_0^d U'' \frac{|g_n|^2 \, dz}{\left(\frac{dj_n}{dt}\right)^2 + (kUj_n)^2} = 0 \qquad \text{...(7)}$$

that is, if there is instability, somewhere in the interval, $U'' = 0$ and changes sign. This was the generalisation obtained by Conrad and Criminale (1965).

Now taking the real part of eqn. (6) we have

$$-\int_0^d (|g_n'|^2 + k^2 |g_n|^2) \, dz$$

$$= k^2 \int_0^d U'' \frac{|g_n|^2 |j_n|^2 U}{\left(\frac{dj_n}{dt}\right)^2 + (kUj_n)^2} dz. \qquad \text{...(8)}$$

Therefore when there is instability, eqn. (7) gives

$$k^2 j_n^2 \int_0^d \frac{|g_n|^2 U''(U - U_I)}{\left(\frac{dj_n}{dt}\right)^2 + (kUj_n)^2} dz$$

$$= -\int_0^d (|g_n'|^2 + k^2 |g_n|^2) \, dz < 0 \qquad \text{...(9)}$$

$U_I(t)$ being the value of $U(z, t)$ at the point where $U'' = 0$.

Hence we get the stronger necessary condition for instability that for time-dependent flow profiles whose functional dependence is separable, $U''(U - U_I) < 0$ somewhere in the field of flow, that is, the absolute value of the vorticity of the basic flow should somewhere have a maximum in the domain of flow.

3. Axisymmetric Parallel Flows

Rayleigh (1892) found a necessary condition for inertial instability of axisymmetric jets, analogous to there being a point of inflection in the velocity profile of a plane parallel basic flow. An improvement of this condition, analogous to Fjørtoft-Høiland condition, was subsequently given by Batchelor and Gill (1962). It is the purpose of this section to generalize these criteria to the class of unsteady axisymmetric parallel flows.

Using cylindrical coordinates (r, θ, z), we take the basic velocity as

$$(0, 0, W(r, t)) \quad (r_1 \leqslant r \leqslant r_2).$$

This represents a jet between the rigid co-axial cylinders $r = r_1, r_2$ where r_1 may be zero and r_2 may be infinite, with z along the axis of the jet. We superimpose on this basic flow a non-axisymmetric disturbance characterized by the pressure π and the velocity components $u_r(r, \theta, z, t)$, $u_\theta(r, \theta, z, t)$ and $u_z(r, \theta, z, t)$. The equations of motion may now be linearized to give the perturbation equations

$$\frac{\partial u_r}{\partial t} + W\frac{\partial u_r}{\partial z} = -D\pi$$

$$\frac{\partial u_\theta}{\partial t} + W\frac{\partial u_\theta}{\partial z} = -\frac{1}{r}\frac{\partial \pi}{\partial \theta}$$

$$\frac{\partial u_z}{\partial t} + W\frac{\partial u_z}{\partial z} + u_r DW = -\frac{\partial \pi}{\partial z}$$

$$D(ru_r) + \frac{\partial u_\theta}{\partial \theta} + r\frac{\partial u_z}{\partial z} = 0$$

where D denotes differentiation with respect to r.

Let $(u_r, u_\theta, u_z, \pi) = \mathrm{Re}\,[iu(r, t),\ v(r, t),\ w(r, t),\ p(r, t)]\, e^{i(k\theta + \alpha z)}$, where α is a real wavenumber and k any integer which represents the azimuthal Fourier component. Then the linearized equations reduce to a form, which on elimination of v, w, and p gives the single linear partial differential equation for u :

$$\left(\frac{\partial}{\partial t} + i\alpha W\right)\left[-u + D\left\{\frac{rD(ru)}{k^2 + \alpha^2 r^2}\right\}\right] - i\alpha r u D\left(\frac{rDW}{k^2 + \alpha^2 r^2}\right) = 0. \quad \ldots(10)$$

We assume a separable solution of the form

$$u(r, t) = \sum_{n=1}^{\infty} g_n(r)\, j_n(t).$$

This assumption is only valid if the base flow $W(r, t)$ is separable and we make such an assumption. We now get equations of the form

$$\left(\frac{dj_n}{dt} + i\alpha W j_n\right)\left[-g_n + D\left\{\frac{rD(rg_n)}{k^2 + \alpha^2 r^2}\right\}\right] - i\alpha r j_n g_n D\left(\frac{rDW}{k^2 + \alpha^2 r^2}\right) = 0. \quad \ldots(11)$$

There is no loss of generality in assuming j_n to be real but g_n to be complex. Hence writing $Q = \frac{r(DW)}{(k^2 + \alpha^2 r^2)}$ we have

$$D\left\{\frac{rD(rg_n)}{k^2+\alpha^2r^2}\right\} - g_n = i\alpha r j_n g_n DQ/(dj_n/dt + i\alpha W j_n). \qquad \text{...(12)}$$

The boundary conditions are that the normal velocity u_r vanishes on the coaxial cylinders $r = r_1, r_2$. For our separable solutions these conditions become

$$g_n = 0 \quad (r = r_1, r_2). \qquad \text{...(13)}$$

These conditions also hold when $r_1 = 0$ (except when $k = 0$) and $r_2 = \infty$. Now, multiplying eqn. (12) by rg_n^* and integrating from r_1 to r_2, using the boundary conditions (13) and clearing the denominator of the right-hand integral of complex numbers, we get

$$-\int_{r_1}^{r_2} r\left(|g_n|^2 + \frac{|D(rg_n)|^2}{k^2+\alpha^2r^2}\right)dr$$

$$= i\alpha j_n \int_{r_1}^{r_2} r^2 |g_n|^2 DQ \frac{\frac{dj_n}{dt} - i\alpha W j_n}{\left(\frac{dj_n}{dt}\right)^2 + (\alpha W j_n)^2} dr. \qquad \text{...(14)}$$

The imaginary part of this expression gives

$$i\alpha j_n \frac{dj_n}{dt} \int_{r_1}^{r_2} \frac{r^2 |g_n|^2 DQ}{\left(\frac{dj_n}{dt}\right)^2 + (\alpha W j_n)^2} dr = 0. \qquad \text{...(15)}$$

Thus either $j_n = 0$ or $dj_n/dt = 0$ or somewhere in the interval $DQ = 0$ and changes sign, that is, for instability, W has a point of inflection with respect to the variable $\rho = k^2 \log r + \frac{1}{2}\alpha^2 r^2$.

A further necessary condition can be found by taking the real part of eqn. (14) so that

$$-\int_{r_1}^{r_2} r\left(|g_n|^2 + \frac{|D(rg_n)|^2}{k^2+\alpha^2r^2}\right)dr = \alpha^2 j_n^2 \int_{r_1}^{r_2} \frac{r^2 |g_n|^2 WDQ}{\left(\frac{dj_n}{dt}\right)^2 + (\alpha W j_n)^2} dr.$$

...(16)

Therefore when instability exists, eqn (15) gives

$$\alpha^2 j_n^2 \int_{r_1}^{r_2} \frac{r^2 \mid g_n \mid^2 (W - W_I)\, DQ}{\left(\frac{dj_n}{dt}\right)^2 + (\alpha W j_n)^2}\, dr = - \int_{r_1}^{r_2} r \left(\mid g_n \mid^2 + \frac{\mid D(r g_n) \mid^2}{k^2 + \alpha^2 r^2} \right) dr < 0 \qquad \text{...(17)}$$

$W_I(t)$ being the value of $W(r, t)$ at $r = r_I$ where $DQ = 0$. Equation (17) gives the stronger necessary condition for instability: if $(W - W_I)\, DQ$ is negative over an appreciable region of the interval $r_1 \leqslant r \leqslant r_2$, then a time-dependent axisymmetric parallel flow can become unstable provided its functional dependence is separable.

4. Spatially Growing Disturbances

Not only most fluid oscillations, in reality, have an amplitude that is growing in some spatial direction, remaining constant with time but also the usual experimental procedure, using a vibrating band for damping or growing disturbances, adopted in verifying the theoretically predicted instability in parallel flow, conform to a situation in which the oscillations are growing or damping in the streamwise direction. In view of the importance of spatially growing theory, we consider an extension of the point-of-inflection criterion to time-dependent flows and spatially varying disturbances.

We start with eqns. (1) and (2) where k is now allowed to be complex. Assuming separable solutions of the form (4), multiplying eqn. (5) by g_n^*, integrating it from 0 to d and using boundary conditions (3), we get

$$\int_0^d \left\{ \mid g_n' \mid^2 + k^2 \mid g_n \mid^2 + \frac{i k j_n U'' \mid g_n \mid^2}{\frac{dj_n}{dt} + i k j_n U} \right\} dz = 0. \qquad \text{...(18)}$$

Taking $k = k_r + i k_i$ in (18), we get the imaginary part as

$$\int_0^d \left\{ 2 k_r k_i \mid g_n \mid^2 + \frac{k_r \left(\frac{dj_n}{dt}\right) U''}{\left(\frac{dj_n}{dt} - k_i j_n U\right)^2 + (k_r j_n U)^2} \mid g_n \mid^2 \right\} dz = 0. \qquad \text{...(19)}$$

If $k_i > 0$ and $dj_n/dt < 0$ (that is, temporally damped disturbances are amplified streamwise) or $k_i > 0$ and $dj_n/dt < 0$ (that is, streamwise damped disturbances are amplified temporally), both conditions leading to instability, it follows that $U'' > 0$ for some point in the interval $0 < z < d$.

Also if $k_i > 0$ and $dj_n/dt < 0$ (that is, disturbances are damped both temporally and spatially), this condition obviously leading to stability, it follows that $U'' < 0$ for some point in the interval.

Hence we can state that for time-dependent flow profiles whose functional dependence is separable, a necessary condition for instability with respect to temporally damped or streamwise damped disturbances is that the velocity profile $U(z, t)$ should have a point with concave-curvature with respect to z in the interval $(0, d)$. Also a necessary condition for stability of such profiles is that $U(z, t)$ should have a point with convex curvature with respect to z in the interval $(0, d)$. This extends a criterion due to Freymuth (1966) for instability of time-independent flows to spatially growing disturbances.

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