A SUFFICIENT CONDITION FOR THE EXISTENCE OF HIGH-DENSITY-BURST CORRECTING LINEAR CODES

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Sometimes, in a communication system, errors occur, of course in the form of a burst but not all the digits inside the burst get disrupted. So it is always useful to study the error correction capabilities of burst codes with weight constraints. In this paper, we have derived a sufficient condition for the existence of an (n, k) linear code which is capable of correcting all single bursts of length b or less that are of weight w or more. Such bursts may be called high-density-bursts. As a special case, taking $b = n$, we may think of codes which for some t, correct all t or more errors and no others. Such codes may be called anti-perfect codes. Existence of such codes in the binary case for $t = n - 1$ has been shown and a relation between these codes and Hamming codes is given. An application of these anti-perfect codes has also been discussed.

1. Introduction

The study of burst error correcting codes has been made with a view to correct errors amongst a specific number of consecutive positions. There are many situations where errors occur, of course, in the form of a burst but not all the digits inside the burst get disrupted. Such a situation can be studied by imposing a suitable weight constraint over the detectable/correctable bursts. An attempt in this direction to find codes capable of detecting and correcting low-density-bursts would find references in papers by Sharma and Dass (1974) and Dass (1974, 1975).

In this paper, we deal with a related problem, i.e., the problem of correcting bursts of length b or less that are of weight w or more ($w \leqslant b$). Such errors may be called high-density-bursts. A sufficient condition for the existence of linear codes which are capable of correcting such errors has been given in Section 2.

As a special case, by taking $b = n$, we have discussed linear codes which for some t correct all t and more errors and no others. We call such codes as 'anti-perfect codes' as the character of such codes is directly opposite to that of perfect codes. In the last section, we have constructed a special type of binary linear codes which correct all $n - 1$ and n errors and no others. A relation between these codes and the well-known Hamming codes has also been discussed.

2. A Sufficient Condition

We start by stating a lemma which is easy to prove.

Lemma 2.1 — If $B(n, b, w)$ denotes the number of bursts of length b or less with weight w or more ($w \leqslant b$) over the space of all n-tuples over $GF(q)$, then

$$B(n, b, w) = \mu + J(n, b, w) \quad \ldots(2.1)$$

where

$$J(n, b, w) = \sum_{i=c}^{b} \left[(n - i + 1) \sum_{j=w-2}^{b-2} \binom{i-2}{j} (q-1)^{j+2} \right]$$

where

$$c = \max\{2, w\}$$

and

$$\mu = \begin{cases} 1 + n(q-1), & \text{if } w = 0 \\ n(q-1), & \text{if } w = 1 \\ 0, & \text{if } w \geqslant 2. \end{cases}$$

The following theorem is an easy application of the above lemma.

Theorem 2.1 — For an (n, k) linear code that corrects all bursts of length b or less with weight w or more ($w \leqslant b$), we must have

$$q^{n-k} \geqslant p + B(n, b, w) \quad \ldots(2.2)$$

where

$$p = \begin{cases} 0, \text{ if } w = 0 \\ 1, \text{ if } w > 0 \end{cases}$$

and $B(n, b, w)$ is given by Lemma 2.1.

In the next theorem, we give a sufficient condition for the existence of a high-density-burst correcting linear code. The proof of the theorem is omitted as it can be derived on the lines of Dass (1975) with suitable modifications.

Theorem 2.2 — Given non-negative integers w and b ($w \leqslant b$), a sufficient condition for the existence of an (n, k) linear code capable of correcting all single bursts of length b or less that are of weight w or more, is

$$q^{n-k} > [M(q) + N(q) + B(n-b, b^*, \bar{w})\, B(b-1, b-1, w-1)] \quad \ldots(2.3)$$

where

$$\bar{w} = \max\{1, w\}, \;\; b^* = \max\{n - b, b\},$$

$B(n, b, w)$ is given by Lemma 2.1 and $M(q)$ and $N(q)$ are given as follows :

$$N(q) = \sum_{i=0}^{p} \binom{b-1}{i} + \sum_{i=\alpha}^{b-1} \binom{b-1}{i}, \text{ if } q = 2$$

where p is the largest odd integer satisfying

$$\left[\frac{p}{2}\right] \leqslant b - w - 1, \alpha = \max\{p + 1, w - 1\}$$

and $[x]$ denoting the integral part of x. Also

$$N(q) = q^{b-1} \text{ if } q \geqslant 3.$$

Also if $q \geqslant 3$, then

$$M(q) = \sum_{k=0}^{b-2} \Bigg[\sum_{r_1=w-1}^{b-k-2} \binom{b-k-2}{r_1} (q-1)^{r_1+1} \Bigg] \times \Bigg[\sum_{\substack{r_2, r_3 \\ r_2+r_3 \leqslant w-2}} \binom{b-k-2}{r_3}\binom{k+1}{r_2} (q-1)^{r_2+r_3} \Bigg]$$

$$+ \sum_{k=0}^{b-2} \Bigg[\sum_{\substack{r_1=w-1 \\ -(k+1)}}^{w-2} \binom{b-k-2}{r_1} (q-1)^{r_1+1} \Bigg] \times \Bigg[\sum_{r_2=0}^{k+1} \binom{k+1}{r_2} (q-1)^{r_2} \Bigg] \times \Bigg[\sum_{\substack{r_3=w-1 \\ -(k+1)}}^{b-k-2} \binom{b-k-2}{r_3} (q-1)^{r_3} \Bigg],$$

and if $q = 2$, then the r_i satisfy the additional constraints :

If $w - 1 - (k + 1) \leqslant r_1 \leqslant w - 2$, then only those values of r_2 and r_3 are permissible for which there exists a positive integer p ($\leqslant r_2$) such that

$$r_1 + (k + 1 - r_2) + p \geqslant w - 1$$

and

$$r_3 + (k + 1 - r_2) + (r_2 - p) \geqslant w - 1.$$

It can be proved that this result turns out to be a generalization of the result obtained by Campopiano given in Theorem 4.10 of Peterson and Weldon (1972).

3. Anti-Perfect Codes

Whenever one obtains a bound, it is desirable to check the suitable values of the parameters for which the bound is tight. In the binary case, for $t = n - 1$ we shall show that the bound obtained in Theorem 2.1 may be used to derive a class of anti-perfect codes. As proposed earlier, by an anti-perfect code we mean a code which for some t corrects all t and more errors and no others.

An expression determining n, the code length, and k, the number of information digits, may be obtained for such codes over $GF(q)$ by putting $w = n - 1$ and $b = n$ in (2.2) in the case of equality. Therefore, an anti-perfect code for $t = n - 1$ will exist if it corrects all $n - 1$ and n errors and satisfies the equation

$$2^{n-k} = n + 2 \qquad \ldots(3.1)$$

i.e. whenever n is 2 less than a power of 2, we may get $(n - 1)$ and n error correcting anti-perfect codes.

To illustrate the idea fully, we discuss two examples. Consider a (6, 3) code whose parity check matrix is given by

$$H = \begin{bmatrix} 0\,0\,0\,1\,1\,1 \\ 0\,1\,1\,0\,0\,1 \\ 1\,0\,1\,0\,1\,0 \end{bmatrix}$$

It can be seen that the syndromes of all the error patterns of weight 5 and 6 are distinct. Therefore, this code may be used for correcting these errors. Further, all the 3-tuples have been accounted for as syndromes, so it gives an anti-perfect code for $t = 5$.

The next higher admissible value of (n, k) satisfying (3.1) is (14, 10). Consider the parity check matrix H given by

$$H = \begin{bmatrix} 1\,1\,1\,0\,1\,1\,1\,0\,0\,0\,1\,0\,0\,0 \\ 1\,1\,0\,1\,1\,0\,0\,1\,0\,1\,0\,1\,0\,0 \\ 1\,0\,1\,1\,0\,0\,1\,1\,1\,0\,0\,0\,1\,0 \\ 0\,1\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,0\,1 \end{bmatrix}$$

This gives rise to a (14, 10) linear code. Also the syndromes of all patterns of weight 13 and 14 being different and exhaustive, we conclude that this code is also anti-perfect for $t = 13$.

In fact for every set of values of n and k satisfying (3.1) such a code exists. This can be established with the help of the following relation between these codes and the Hamming's single-error-correcting binary perfect codes.

It can be proved with a little effort that if we drop any column from the parity check matrix of a Hamming code, the resultant matrix gives rise to an anti-perfect code with same number of information digits but with length one less than that of the corresponding Hamming code.

The existence of Hamming codes for every set of values of n and k satisfying

$$2^{n-k} = n + 1 \qquad \ldots(3.2)$$

thus ensures the existence of anti-perfect codes for all values of n and k satisfying

$$2^{n-k} = n + 2.$$

A Decoding Process

It is interesting to note that if we choose the parity check matrix H of an anti-perfect code in such a way that its columns are the binary representations of their positions then the syndrome of an error-pattern of weight $(n - 1)$ is same as that of an error-pattern of weight one obtained from it by interchanging 0 and 1 and reversing its order, i.e. changing $(a_1, a_2, \ldots, a_n)$ to

$$(1 - a_n, 1 - a_{n-1}, \ldots, 1 - a_2, 1 - a_1).$$

By choosing the columns of the parity check matrix H in the above mentioned way, the decoding process of anti-perfect codes becomes very simple. If the syndrome of a received pattern coincides with the ith column of H, this means that all the digits in the received pattern except the $(n - i)$th digit, are in error. If the syndrome does not coincide with any of the columns of H, this means that all the digits are in error.

An Application

In a binary channel, suitable for tolerating single errors, if all the arrangements get reversed and their restructuring becomes either cumbersome or expansive then the codes discussed above may be employed.

It may be pointed out that the existence of such anti-perfect codes for other values of t and q is not yet known.

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REFERENCES

Dass, B. K. (1974). Bounds for codes detecting burst-errors with weight constraint. *Proc Conf. on Measures of Information and Their Applications, I.I.T., Bombay*, August 16-18.

——— (1975). A sufficient bound for codes correcting bursts with weight constraint. *J. ACM*, 501-507.

Peterson, W. W., and Weldon, E. J. (Jr.) (1972). Error Correcting Codes. John Wiley M.I.T. Press.

Sharma, B. D., and Dass, B. K. (1974). Extended Varsharmov Gilbert and sphere pack bounds for burst codes. *IEEE Trans.*, IT-20, 291-92.