NOTE ON AN INTEGRAL TRANSFORM

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An integral transform with Legendre function for a broken interval as its Kernel has been derived from boundary value problem and its Perseval relation has also been obtained. Finally a particular integral is evaluated.

Mehler (1881) has developed integral transforms involving Legendre function as the Kernel with inverse transform containing associated Legendre function $P_\nu^\mu(x)$. Transforms containing integrals with respect to the degree of Legendre functions may also occur in the boundary value problems involving conical boundaries. Here we consider integration over the degree of the Legendre function. In the solution of such boundary value problems, the integrals involving both order and degree may occur, but we should use the formula which gives quicker convergence.

We have a method of representing Dirac δ-function (cf. Friedmann 1956), with the help of two solutions of a Sturm-Liouville type of differential equation of second order, and from this Felsen (1958) has obtained a unique δ-function representation in terms of an integral in which the degree of the Legendre function is the variable and consequently a transform is obtained. Naylor (1963) on the other hand, has obtained some transforms with Bessel function as the Kernel, from some boundary value condition. Mandal (1971) has also obtained a transform associated with integration with respect to degree of Legendre's associated function obtained from a boundary value problem for a complete interval. In this paper, a more general transform has been derived for a broken interval.

Lowndes (1964) obtained Perseval relations in simple forms for the Lebedev Kontrovitch transforms and Mehler-Fok transforms, and obtained several integrals by using the Perseval relation. Mandal (1971) has also obtained 'Perseval relations' for the transform developed by him and obtained a particular Integral involving Meijer G-function. Here also from the transform developed, Perseval relation has been established and the value of a particular integral has been obtained.

The Laplace's equation $\nabla^2 u = 0$ in the region $0 \leqslant r < \infty$, $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \varphi \leqslant \alpha$ reduces to

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$$s(s+1)\,v + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial v}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial\varphi^2} = 0 \qquad \ldots(1)$$

where $u = vr^{-s-1}$, Re $s > -1$ taking $u \to 0$ as $r \to \infty$.

The boundary conditions which $v(\theta, \varphi)$ satisfies are

$$v = f(\theta), \quad \varphi = 0, \quad 0 \leqslant \theta \leqslant \alpha < \pi$$
$$v = 0, \quad \varphi = \beta, \quad 0 \leqslant \theta \leqslant \alpha < \pi.$$

Let

$$v_n = \int_0^\beta v \sin\lambda\varphi\, d\varphi, \text{ where } \lambda = \frac{n\pi}{\beta}, \; v = \frac{2}{\beta}\sum_{n=1}^{\infty} v_n(\theta)\sin\lambda\varphi.$$

From (1), we see that $v_n(\theta)$ satisfies the equation

$$\frac{d}{d\theta}\left(\sin\theta\,\frac{d\psi}{d\theta}\right) + \left[s(s+1)\sin\theta - \frac{\lambda^2}{\sin\theta}\right]\psi = -\frac{\lambda f(\theta)}{\sin\theta}. \qquad \ldots(2)$$

The solution of (2) is $\psi(\theta) = -\lambda\int_0^\alpha \frac{f(\theta')}{\sin\theta'}\,G(\theta,\theta')\,d\theta$ $\qquad \ldots(3)$

where $G(\theta, \theta')$ is the Green's function corresponding to eqn. (2) and is given when $G = 0$ on $\theta = 0$ and $\theta = \alpha$,

$$G(\theta,\theta') = -\frac{\pi}{2}\frac{\Gamma(s+1+\lambda)}{\Gamma(s+1-\lambda)}\frac{P_s^{-\lambda}(\cos\theta)}{\sin(s+1-\lambda)\pi}$$
$$\times\frac{\left[P_s^{-\lambda}(\cos\theta)\cdot P_s^{-\lambda}(-\cos\alpha) - P_s^{-\lambda}(-\cos\theta)\cdot P_s^{-\lambda}(\cos\alpha)\right]}{P_s^{-\lambda}(\cos\alpha)},$$

$$0 < \theta < \theta' < \alpha. \qquad \ldots(4a)$$

To write Green's function for the region $0 < \theta' < \theta < \alpha$, θ and θ' are to be interchanged. Now G can be written as

$$G(\theta,\theta') = \frac{1}{2i}\int_L \frac{\Gamma(s+1+\mu)}{\Gamma(s+1-\mu)}\frac{P_s^{-\mu}(\cos\theta)}{\sin(s+1-\mu)\pi}$$
$$\times\frac{\left[P_s^{-\lambda}(\cos\theta)\cdot P_s^{-\lambda}(-\cos\alpha) - P_s^{-\lambda}(-\cos\theta)\,P_s^{-\lambda}(\cos\alpha)\right]}{P_s^{-\lambda}(\cos\alpha)}$$
$$\times\frac{\mu\,d\mu}{\mu^2-\lambda^2} \qquad \ldots(4b)$$

where L is the straight line $\mathrm{Re}\,\mu = c$, $-\lambda < s < c < \lambda$ (taking $|\mathrm{Re}\, s| < \lambda$). To show this, we observe that the integrand has singularities only at $\mu = \pm\lambda$. When $\mathrm{Re}\,\mu$ is large, the integrand is

$$\approx \frac{1}{2\mu} e^{\mu\gamma} - \frac{1}{2\mu} e^{\mu\delta}$$

where

$$\gamma = \ln\left(\frac{\tan\frac{\theta}{2}}{\tan\frac{\theta'}{2}}\right) \quad \text{and} \quad \delta = \ln\left(\frac{\tan\frac{\theta}{2}\,.\,\tan\frac{\theta'}{2}}{\tan\frac{\alpha}{2}\,.\,\tan\frac{\alpha}{2}}\right)$$

and when $\theta < \theta' < \alpha$ both γ and δ are negative, so that the integrand vanishes as $|\mu| \to \infty$, $\mathrm{Re}\,\mu > 0$. Again if $\theta' < \theta < \alpha$, θ and θ' are interchanged and we have then

$$\gamma = \ln\frac{\tan\frac{\theta'}{2}}{\tan\frac{\theta}{2}} \quad \text{and} \quad \delta = \ln\frac{\tan\frac{\theta'}{2}\,.\,\tan\frac{\theta}{2}}{\tan\frac{\alpha}{2}\,.\,\tan\frac{\alpha}{2}}$$

so that both γ and δ are again negative. Hence in both the cases the integrand tends to zero as $|\mu| \to \infty$, $\mathrm{Re}\,\mu > 0$.

Now, taking a large semicircle in the positive μ plane with L as its diameter, the only singularity of the integrand is at $\mu = \lambda$, and we see that the integrand in (4b) reduces to (4a). So (4a) is written in terms of a contour integral in (4b).

Substituting in (3) from (4b)

$$v_n(\theta) = -\frac{\lambda}{2i}\int_L \frac{\Gamma(s+1+\mu)}{\Gamma(s+1-\mu)}\,\frac{P_s^{-\mu}(\cos\theta)}{\sin(s+1-\mu)\,\pi}\,F(\mu)\,\frac{\mu\,d\mu}{\mu^2-\lambda^2}$$

where

$$F(\mu) = \int_0^\alpha \frac{f(\theta')}{\sin\theta'}$$

$$\times \frac{\left[P_s^{-\mu}(\cos\theta')\cdot P_s^{-\mu}(-\cos\alpha) - P_s^{-\mu}(-\cos\theta')\cdot P_s^{-\mu}(\cos\alpha)\right]}{P_s^{-\mu}(\cos\alpha)}\,d\theta'.$$

Hence

$$v(\theta, \varphi) = \frac{2}{\alpha}\sum_1^\infty v_n \sin\lambda\varphi.$$

But

$$\frac{\sin \mu(\alpha - \varphi)}{\sin \mu\alpha} = -\frac{2}{\alpha} \sum_{1}^{\infty} \frac{\lambda \sin \lambda\varphi}{\mu^2 - \lambda^2}.$$

Therefore

$$v(\theta, \varphi) = \frac{1}{2i} \int_L \frac{\Gamma(s + 1 + \mu)\ P_s^{-\mu}(\cos\theta)}{\Gamma(s + 1 - \mu) \sin(s + 1 - \mu)\pi} \frac{\sin \mu(\alpha - \varphi)}{\sin \mu\alpha} F(\mu)\ \mu\ d\mu.$$

As $v(\theta, \varphi) = f(\theta)$ when $\varphi = 0$, we arrive at by putting $\varphi = 0$

$$f(\theta) = \frac{1}{2\pi i} \int_L \Gamma(s + 1 + \mu)\ \Gamma(\mu - s)\ P_s^{-\mu}(\cos\theta)\ F(\mu)\ \mu\ d\mu.$$

where L is Re $\mu = c,\ -\lambda < s < c < \lambda$. As Re $s > -1$ and $\lambda = \frac{n\pi}{\beta}\ (n = 1, 2, \ldots)$, we may take $c \to 0$ so that

$$f(\theta) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(s + 1 + \mu)\ \Gamma(\mu - s)\ P_s^{-\mu}(\cos\theta)\ F(\mu, s)\ \mu\ d\mu \qquad \ldots(5)$$

where

$$F(\mu, s) = \int_0^{\alpha} \frac{f(\theta)}{\sin\theta} \times \frac{\left[P_s^{-\mu}(\cos\theta) \cdot P_s^{-\mu}(-\cos\alpha) - P_s^{-\mu}(-\cos\theta) \cdot P_s^{-\mu}(\cos\alpha)\right]}{P_s^{-\mu}(\cos\alpha)}\, d\theta. \qquad \ldots(6)$$

So, (6) gives the transform and (5) the inverse.

Let $F(\mu, s)$ and $G(\mu, s)$ be the transforms of $f(\theta)$ and $g(\theta)$ respectively. To obtain Perseval relation, we write

$$\int_{-i\infty}^{i\infty} \mu\psi(\mu, s)\ F(\mu, s)\ G(\mu, s)\ d\mu = \int_{-i\infty}^{i\infty} \mu\psi(\mu, s)\ F(\mu, s) \int_{\pi-\alpha}^{\pi} \frac{g(\pi - \theta)}{\sin\theta} \times \frac{\left[P_s^{-\mu}(\cos\theta) \cdot P_s^{-\mu}(-\cos\alpha) - P_s^{-\mu}(-\cos\theta) \cdot P_s^{-\mu}(\cos\alpha)\right]}{P_s^{-\mu}(\cos\alpha)}\, d\theta. \qquad \ldots(7)$$

where $\psi(\mu, s)$ is to be determined. Assuming the interchange of the order of integration to be possible, we get

$$= -\int_{\pi-\alpha}^{\pi} \frac{g(\pi-\theta)}{\sin\theta}\, d\theta \int_{-i\infty}^{i\infty} \mu\psi(\mu, s)\, F(\mu, s)\, P_s^{-\mu}(\cos\theta)\, d\mu$$

$$+\int_{\pi-\alpha}^{\pi} \frac{g(\pi-\theta)}{\sin\theta}\, d\theta \int_{-i\infty}^{i\infty} \mu\psi(\mu, s)\, F(\mu, s)\, P_s^{-\mu}(\cos\theta)\, \frac{P_s^{-\mu}(-\cos\alpha)}{P_s^{-\mu}(\cos\alpha)}\, d\mu. \quad \ldots(8)$$

Now taking $\psi(\mu, s)$ as the expression $\frac{1}{2\pi i}\Gamma(s+1+\mu)\,\Gamma(\mu-s)$ we get

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \mu\Gamma(s+1+\mu)\,\Gamma(\mu-s)\, F(\mu, s)\, G(\mu, s)\, d\mu$$

$$= \int_0^{\alpha} \frac{1}{\sin\theta}\, \frac{P_s^{-\mu}(-\cos\alpha)}{P_s^{-\mu}(\cos\alpha)}\, f(\theta)\, g(\theta)\, d\theta - \int_{\pi-\alpha}^{\pi} \frac{1}{\sin\theta}\, g(\pi-\theta)\, f(\theta)\, d\theta.$$

$$\ldots(9)$$

As a particular case, when $\alpha = \pi$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \mu\Gamma(s+1+\mu)\,\Gamma(\mu-s)\, F(\mu, s)\, G(\mu, s)\, d\mu = -\int_0^{\pi} \frac{g(\pi-\theta)}{\sin\theta}\, f(\theta)\, d\theta$$

which is established in (6).

We now define a particular integral with the Perseval relation, when $\alpha = \frac{\pi}{2}$
If $f(\theta) = (\cos\theta)^{\sigma}\,(\sin\theta)^{3/2}$, then

$$F(\mu, s) = \frac{2^{\mu-1}\Gamma\left(\frac{3+2\mu}{4}\right)\Gamma\left(\frac{1}{2}+\frac{\sigma}{2}\right)}{\Gamma(1-\mu)\,\Gamma\left(\frac{5}{4}+\frac{\sigma}{2}-\frac{\mu}{2}\right)}$$

$$\times\left[1+\frac{\sin\pi(s-\mu)}{\sin\pi\mu}\,\frac{\Gamma(s-\mu+1)}{\Gamma(s+\mu+1)}\right]$$

$$\times\ {}_3F_2\left(\frac{s-\mu+1}{2}, \frac{s-\mu}{2}, 1-\mu, \frac{2\sigma-2\mu+5}{4}, 1\right)+$$

(*equation continued on p.* 826)

$$+ \frac{2^{-\mu-1}\Gamma\left(\frac{3-2\mu}{4}\right)\Gamma\left(\frac{1}{2}+\frac{\sigma}{2}\right)}{\Gamma(1+\mu)\,\Gamma\left(\frac{5}{4}+\frac{\sigma}{2}+\frac{\mu}{2}\right)}$$

$$\times \frac{\sin\pi(s-\mu)}{\sin\frac{\pi}{4}}\frac{\Gamma(s-\mu+1)}{\Gamma(s+\mu+1)}$$

$$\times\, {}_3F_2\left(\frac{s+\mu+1}{2}, \frac{s+\mu}{2}, \frac{3+2\mu}{4}, 1+\mu, \frac{2\sigma+2\mu+5}{4}, 1\right). \qquad \ldots(10)$$

Takin $g(\theta) = (\cos\theta)^{\sigma'}(\sin\theta)^{3/2}$,

$$G(\mu, s) = \frac{2^{\mu-1}\Gamma\left(\frac{3+2\mu}{4}\right)\Gamma\left(\frac{1}{2}+\frac{\sigma'}{2}\right)}{\Gamma(1-\mu)\,\Gamma\left(\frac{5}{4}+\frac{\sigma'}{2}-\frac{\mu}{2}\right)}$$

$$\times\left[1+\frac{\sin\pi(s-\mu)}{\sin\pi\mu}\frac{\Gamma(s-\mu+1)}{\Gamma(s+\mu+1)}\right]$$

$$\times\, {}_3F_2\left(\frac{s-\mu+1}{2}, \frac{s-\mu}{2}, 1-\mu, \frac{2\sigma'-2\mu+5}{4}, 1\right)$$

$$+ \frac{2^{-\mu-1}\Gamma\left(\frac{3-2\mu}{4}\right)\Gamma\left(\frac{1}{2}+\frac{\sigma'}{2}\right)}{\Gamma(1+\mu)\,\Gamma\left(\frac{5}{4}+\frac{\sigma'}{2}+\frac{\mu}{2}\right)}$$

$$\times \frac{\sin\pi(s-\mu)}{\sin\frac{\pi}{4}}\frac{\Gamma(s-\mu+1)}{\Gamma(s+\mu+1)}$$

$$\times\, {}_3F_2\left(\frac{s+\mu+1}{2}, \frac{s+\mu}{2}, \frac{3+2\mu}{4}, 1+\mu, \frac{2\sigma'+2\mu+5}{4}, 1\right). \qquad \ldots(11)$$

Then the integral

$$I = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \mu\Gamma(s+1+\mu)\,\Gamma(\mu-s)\,F(\mu, s)\,G(\mu, s)\,d\mu$$

$$= [1-(-1)^{2\sigma+\sigma'}]\,\frac{\Gamma\left(\frac{\sigma+\sigma'+1}{2}\right)}{2\Gamma\left(\frac{\sigma+\sigma'+1}{2}+1\right)}$$

$$= [1-(-1)^{2\sigma+\sigma'}]\cdot\frac{1}{\sigma+\sigma'+1}\,. \qquad \ldots(12)$$

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