ON LIFTS OF ALMOST QUATERNION STRUCTURE

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In this paper we obtain conditions under which horizontal and complete lifts of tensor fields in M admitting almost quaternion structure $\{F, G, H\}$ may define a similar structure in ${}^cT(M)$, the cotangent bundle of M (Yano and Patterson 1967). Further by introducing a symmetric affine connection in M we obtain equivalent conditions for the set $\{F^c, C^c, H^c\}$ to be almost quaternion. Lastly we obtain a result on the integrability of the almost quaternion structure $\{F^c, G^c, H^c\}$ in ${}^cT(M)$ (Yano and Ako 1972).

Let M be a differentiable manifold of class C^∞ with almost quaternion structure. It is known (Yano 1973) that a manifold admitting such a structure is $4p$-dimensional, p being a positive integer. Let ${}^cT(M)$ be the cotangent bundle of M, i.e. the bundle of covariant vectors in M. ${}^cT(M)$ is also a differentiable manifold of class C^∞ and dimension $8p$.

Let $4p = n$ where n is even $\geqslant 4$.

Therefore $\dim {}^cT(M) = 2n$.

In this paper we consider methods by which tensor fields defining almost quaternion structure in M can be extended to ${}^cT(M)$. These extensions are called lifts of the tensor fields in M. We consider complete and horizontal lifts of tensor fields of type (1, 1) on M and also establish conditions under which these lifts may define an almost quaternion structure, more so an integrable quaternion structure.

§1. We recall that a manifold is said to possess an almost quaternion structure of first kind if ∃ a set of three distinct tensor fields F, G, H of type (1, 1) which satisfy (Yano 1973)

$$F^2 = -1; \qquad G^2 = -1; \qquad H^2 = -1$$

$$F = GH = -HG; \; G = HF = -FH; \; H = FG = -GF. \qquad \ldots(1.1)$$

Similarly a manifold is called an almost quaternion manifold of second kind if for the tensor fields F, G, H of type (1, 1) the following are satisfied :

$$F^2 = -1; \qquad G^2 = 1; \qquad H^2 = 1$$

$$F = GH = -HG; \; G = HF = -FH; \; H = FG = -GF. \qquad \ldots(1.2)$$

If A is a point in M then $\pi^{-1}(A)$ is the fibre over A where $\pi : {}^cT(M) \to M$ is the projection map. Any point $P \in \pi^{-1}(A)$ is an ordered pair (A, p_A) where p is a 1-form and p_A its value at A. If U is a coordinate neighbourhood in M then U induces a coordinate neighbourhood $\pi^{-1}(U)$ in ${}^cT(M)$.

It is known (Yano and Patterson 1967) that the complete lift F^c of a tensor field F of type (1, 1) is a tensor field of the same type whose components $\tilde{F}^A_B$ in $\pi^{-1}(U)$ are given by

$$\tilde{F}^h_i = F^h_i \; ; \; \tilde{F}^h_{\bar{i}} = 0$$

$$\tilde{F}^{\bar{h}}_i = p_a\left(\frac{\partial F^a_h}{\partial x^i} - \frac{\partial F^a_i}{\partial x^h}\right); \; \tilde{F}^{\bar{h}}_{\bar{i}} = F^i_h$$

where $A, B, C, \ldots$ run over $1, 2, \ldots, 2n$,

$a, b, c, h, i, j \ldots$ run over $1, 2, \ldots n$ and $\bar{i} = i + n$ etc.

Thus

$$F^c = \begin{pmatrix} F^h_i & 0 \\ p_a\left(\frac{\partial F^a_h}{\partial x^i} - \frac{\partial F^a_i}{\partial x^h}\right) & F^i_h \end{pmatrix}$$

$$= \begin{pmatrix} F^h_i & 0 \\ p_a \partial_{[i} F^a_{h]} & F^i_h \end{pmatrix}.$$

Again we have (Prakash and Bahadur 1973)

$$(F^c)^2 = \tilde{F}^A_B \tilde{F}^B_C = \begin{pmatrix} + F^i_j F^j_k & 0 \\ M_{ik} & F^j_i F^k_j \end{pmatrix}$$

where

$$M_{ik} = F^j_k \, p_a \, \partial_{[j} F^a_{i]} + F^j_i \, p_s \, \partial_{[k} F^s_{j]} \cdot$$

Since $F^2 = -1$

therefore $$(F^c)^2 = \begin{pmatrix} -\delta^i_k & 0 \\ M_{ik} & -\delta^k_i \end{pmatrix}.$$

Note : We denote $p_a\left(\frac{\partial F^a_h}{\partial x^i} - \frac{\partial F^a_i}{\partial x^h}\right)$ by $p_a \partial_{[i} F^a_{h]}$

Thus $(F^c)^2 = -1$ if $M_{ik} = 0$ i.e.,

$$F^j_k \, p_a \, \partial_{[j} F^a_{i]} + F^j_i \, p_s \, \partial_{[k} F^s_{j]} = 0. \qquad \text{...(1.3)}$$

Similarly $(G^c)^2 = -1$ and $(H^c)^2 = -1$ if

$$G^j_k \, p_a \, \partial_{[j} G^a_{i]} + G^j_i \, p_s \, \partial_{[k} G^s_{j]} = 0 \qquad \text{...(1.4)}$$

and

$$H^j_k \, p_a \, \partial_{[j} H^a_{i]} + H^j_i \, p_s \, \partial_{[k} H^s_{j]} = 0. \qquad \text{...(1.5)}$$

Again

$$G^c = \tilde{G}^A_B = \begin{pmatrix} G^i_j & 0 \\ p_a \, \partial_{[j} G^a_{i]} & G^j_i \end{pmatrix}.$$

Thus $$G^c H^c = \begin{pmatrix} F^i_k & 0 \\ P_{ik} & F^k_i \end{pmatrix}$$

where

$$P_{ik} = p_a \, \partial_{[j} G^a_{i]} H^j_k + G^j_i \, p_s \, \partial_{[k} H^s_{j]} \cdot$$

Thus $G^c H^c = F^c$ if

$$P_{ik} = p_l \, (\partial_{[k} F^l_{i]}) = p_l \left(\frac{\partial F^l_i}{\partial x^k} - \frac{\partial F^l_k}{\partial x^i} \right).$$

On simplifying the above equation becomes

$$p_a(\partial_j \, G^a_i) \, H^j_k + G^j_i \, p_a \, \partial_{[k} H^a_{j]} = p_a(\partial_k \, G^a_j) \, H^j_i + p_a \, G^a_j \, \partial_{[k} H^j_{i]} \cdot \qquad \text{...(1.6)}$$

Similarly we obtain conditions for $F^c G^c = H^c$ and $H^c F^c = G^c$ respectively as:

$$p_a(\partial_j \, F^a_i) \, G^j_k + F^j_i \, p_s \, \partial_{[k} G^s_{j]} = p_a(\partial_k \, F^a_j) \, G^j_i + p_a \, F^a_j \, \partial_{[k} G^j_{i]} \qquad \text{...(1.7)}$$

$$p_a(\partial_j \, H^a_i) \, F^j_k + H^j_i \, p_s \, \partial_{[k} F^s_{j]} = p_a(\partial_k \, H^a_j) \, F^j_i + p_a \, H^a_j \, \partial_{[k} F^j_{i]} \cdot \qquad \text{...(1.8)}$$

Further since

$$H^c G^c = \begin{pmatrix} -F^i_k & 0 \\ p_s \, \partial_{[j} H^s_{i]} G^j_k + H^j_i \, p_a \, \partial_{[k} G^a_{j]} & -F^k_i \end{pmatrix}$$

Thus

$$G^cH^c + H^cG^c = \begin{pmatrix} 0 & 0 \\ p_s\partial_{[j}H^s_{i]}\, G^j_k + H^j_i\, p_a\partial_{[k}G^a_{j]} & 0 \\ + G^j_i\, p_s\partial_{[k}H^s_{j]} + p_a\partial_{[j}G^a_{i]} & H^j_k \end{pmatrix}.$$

From above we obtain

$$G^cH^c + H^cG^c = 0 \text{ if}$$

$$p_s\,\partial_{[j}H^s_{i]}\, G^j_k + H^j_i\, p_a\,\partial_{[k}G^a_{j]} + G^j_i\, p_s\,\partial_{[k}H^s_{j]} + p_a\,\partial_{[j}G^a_{i]}\, H^j_k = 0. \quad \text{...(1.9)}$$

Similarly we obtain conditions for $H^cF^c + F^cH^c = 0$ and $F^cG^c + G^cF^c = 0$ as :

$$p_s\,\partial_{[j}F^s_{i]}\, H^j_k + F^i_i\, p_a\,\partial_{[k}H^a_{j]} + H^j_i\, p_s\,\partial_{[k}F^s_{j]} + p_a\,\partial_{[j}H^a_{i]}\, F^j_k = 0 \quad \text{...(1.10)}$$

and

$$p_s\,\partial_{[j}G^s_{i]}\, F^j_k + G^j_i\, p_a\,\partial_{[k}F^a_{j]} + F^j_i\, p_s\,\partial_{[k}G^s_{j]} + p_a\,\partial_{[j}F^a_{i]}\, G^j_k = 0. \quad \text{...(1.11)}$$

Thus we obtain :

Theorem 1.1 — If a manifold M has almost quaternion structure (F, G, H) of first kind then (F^c, G^c, H^c) will be an almost quaternion structure of first kind in ${}^cT(M)$ if and only if the equations (1.3), (1.4), (1.5), (1.6), (1.7), (1.8), (1.9), (1.10), (1.11) are true.

We obtain a similar theorem for almost quaternion manifold of second kind.

§2. We now express the above equations in terms of connection coefficients as follows :

It is known (Yano 1965)

$$F^a_{h,i} = \frac{\partial F^a_h}{\partial x^i} + \Gamma^a_{li}\, F^l_h - \Gamma^k_{hi}\, F^a_k$$

$$F^a_{i,h} = \frac{\partial F^a_i}{\partial x^h} + \Gamma^a_{lh}\, F^l_i - \Gamma^k_{ih}\, F^a_k$$

where Γ^k_{hi} are the components of an affine connection ∇ in M.

Assuming Γ to be symmetric equations (1.3) to (1.11) can be equivalently expressed as:

$$p_a(F^a_{i,j} - F^a_{j,i})\, F^j_k + 2p_a\,\Gamma^a_l\, F^l_{[i\,j]}\, F^j_k + F^j_i\, p_a\,(F^a_{j,k} - F^a_{k,j})$$
$$+ 2F^j_i\; p_a\; \Gamma^a_j\, F^l_{[j\,k]} = 0 \quad \text{...(2.1)}$$

$$p_a(G^a_{i,j} - G^a_{j,i}) + 2p_a \Gamma^a_l G^l_{[ij]} G^j_k + G^j_i p_a (G^a_{j,k} - G^a_{k,j})$$
$$+ 2G^j_i p_a \Gamma^a_l G^l_{[jk]} = 0 \qquad \ldots(2.2)$$

and

$$p_a(H^a_{i,j} - H^a_{j,i}) H^j_k + 2p_a \Gamma^a_l H^l_{[ij]} H^j_k + H^j_i p_a(H^a_{j,k} - H^a_{k,j})$$
$$+ 2H^j_i p_a \Gamma^a_l H^l_{[jk]} = 0 \qquad \ldots(2.3)$$

$$p_a(F^a_{i,k} - F^a_{k,i}) + 2p_a \Gamma^a_l F^l_{[ik]}$$
$$= p_t(G^t_{i,j} - G^t_{j,i}) H^j_k + 2p_t \Gamma^t_l G^l_{[ij]} H^j_k$$
$$+ G^j_i p_s(H^s_{j,k} - H^s_{k,j}) + 2G^j_i p_s \Gamma^s_l H^l_{[jk]} \qquad \ldots(2.4)$$

$$p_a(H^a_{i,k} - H^a_{k,i}) + 2p_a \Gamma^a_l H^l_{[ik]}$$
$$= p_t(F^t_{i,j} - F^t_{j,i}) G^j_k + 2p_t \Gamma^t_l F^l_{[ij]} G^j_k$$
$$+ F^j_i p_s(G^s_{j,k} - G^s_{k,j}) + 2F^j_i p_s \Gamma^s_l G^l_{[jk]} \qquad \ldots(2.5)$$

and

$$p_a(G^a_{i,k} - G^a_{k,i}) + 2p_a \Gamma^a_l G^l_{[ik]}$$
$$= p_t(H^t_{i,j} - H^t_{j,i}) F^j_k + 2p_t \Gamma^t_l H^l_{[ij]} F^j_k$$
$$+ H^j_i p_s(F^s_{j,k} - F^s_{k,j}) + 2H^j_i p_s \Gamma^s_l F^l_{[jk]} \qquad \ldots(2.6)$$

$$p_s(H^s_{i,j} - H^s_{j,i}) G^j_k + 2p_s \Gamma^s_l H^l_{[ij]} G^j_k + H^j_i p_a(G^a_{j,k} - G^a_{k,j})$$
$$+ 2p_a \Gamma^a_l G^l_{[jk]} H^j_i + G^j_i p_s(H^s_{j,k} - H^s_{k,j})$$
$$+ 2G^j_i p_s \Gamma^s_l H^l_{[jk]} + p_a(G^a_{i,j} - G^a_{j,i}) H^j_k$$
$$+ 2p_a \Gamma^a_l G^l_{[ij]} H^j_k = 0 \qquad \ldots(2.7)$$

$$p_s(F^s_{i,j} - F^s_{j,i}) H^j_k + 2p_s \Gamma^s_l F^l_{[ij]} H^j_k + F^j_i p_a(H^a_{j,k} - H^a_{k,j})$$
$$+ 2F^j_i p_a \Gamma^a_l H^l_{[jk]} + H^j_i p_s(F^s_{j,k} - F^s_{k,j}) + 2H^j_i p_s \Gamma^s_l F^l_{[jk]}$$
$$+ p_a(H^a_{i,j} - H^a_{j,i}) F^j_k + 2p_a \Gamma^a_l H^l_{[ij]} F^j_k = 0 \qquad \ldots(2.8)$$

$$p_s(G^s_{i,j} - G^s_{j,i}) F^j_k + 2p_s \Gamma^s_l G^l_{[i\,j]} F^j_k + G^j_i\, p_a(F^a_{j,k} - F^a_{k,j})$$

$$+ 2G^j_i\, p_a \Gamma^a_l F^l_{[j\,k]} + F^j_i\, p_s(G^s_{j,k} - G^s_{k,j}) + 2F^j_i\, p_s \Gamma^s_l G^l_{[j\,k]}$$

$$+ p_a(F^a_{i,j} - F^a_{j,i}) G^j_k + 2p_a \Gamma^a_l F^l_{[i\,j]} G^j_k = 0. \qquad \ldots(2.9)$$

Thus Theorem 1.1 can be restated as :

Theorem 2.1 — In an almost quaternion manifold M of first kind (respectively second kind) with almost quaternion structure (F, G, H); ${}^cT(M)$ possesses an almost quaternion structure (F^c, G^c, H^c) of first kind (respectively second kind) if and only if eqns. (1.3) to (1.11) or equivalently eqns. (2.1) to (2.9) are true where ∇ is a symmetric affine connection in M.

Suppose

$$p_a(F^a_{h,i} - F^a_{i,h}) = v_{h,i} - v_{i,h} = \text{curl } v$$

$$p_t(G^t_{i,j} - G^t_{j,i}) = \tilde{v}_{i,j} - \tilde{v}_{j,i} = \text{curl } \tilde{v}$$

$$p_s(H^s_{j,k} - H^s_{k,j}) = \tilde{\tilde{v}}_{j,k} - \tilde{\tilde{v}}_{k,j} = \text{curl } \tilde{\tilde{v}}.$$

In case $F^a_{h,i} = G^a_{h,i} = H^a_{h,i} = 0$ eqns. (2.1) to (2.9) become

$$p_a \Gamma^a_l F^l_{[i\,j]} F^j_k + F^j_i\, p_a \Gamma^a_l F^l_{[j\,k]} = 0 \qquad \ldots(2.10)$$

$$p_a \Gamma^a_l G^l_{[i\,j]} G^j_k + G^j_i\, p_a \Gamma^a_l G^l_{[j\,k]} = 0 \qquad \ldots(2.11)$$

$$p_a \Gamma^a_l H^l_{[i\,j]} H^j_k + H^j_i\, p_a \Gamma^a_l H^l_{[j\,k]} = 0 \qquad \ldots(2.12)$$

$$p_a \Gamma^a_l F^l_{[i\,k]} = p_t \Gamma^t_l G^l_{[i\,j]} H^j_k + G^j_i\, p_s \Gamma^s_l H^l_{[j\,k]} \qquad \ldots(2.13)$$

$$p_a \Gamma^a_l H^l_{[i\,k]} = p_t \Gamma^t_l F^l_{[i\,j]} G^j_k + F^j_i\, p_s \Gamma^s_l G^l_{[j\,k]} \qquad \ldots(2.14)$$

$$p_a \Gamma^a_l G^l_{[i\,k]} = p_t \Gamma^t_l H^l_{[i\,j]} F^j_k + H^j_i\, p_s \Gamma^s_l F^l_{[j\,k]} \qquad \ldots(2.15)$$

$$2p_s \Gamma^s_l H^l_{[i\,j]} G^j_k + 2H^j_i\, p_a \Gamma^a_l G^l_{[j\,k]} + 2G^j_i\, p_s \Gamma^s_l H^l_{[j\,k]}$$

$$+ 2p_a \Gamma^a_l G^l_{[i\,j]} H^j_k = 0 \qquad \ldots(2.16)$$

$$2p_s\,\Gamma^s_l\,F^l_{[i\ j]}H^j_k + 2F^j_i\,p_a\,\Gamma^a_l\,H^l_{[j\ k]} + 2H^j_i\,p_s\,\Gamma^s_l\,F^l_{[j\ k]}$$

$$+ 2p_a\,\Gamma^a_l\,H^l_{[i\ j]}F^j_k = 0 \qquad \text{...(2.17)}$$

$$2p_s\,\Gamma^s_l\,G^l_{[i\ j]}F^j_k + 2G^j_i\,p_a\,\Gamma^a_l\,F^l_{[j\ k]} + 2F^j_i\,p_s\,\Gamma^s_l\,G^l_{[j\ k]}$$

$$+ 2p_a\,\Gamma^a_l\,F^l_{[i\ j]}G^j_k = 0. \qquad \text{...(2.18)}$$

Hence we have the following corollaries :

Corollary A — In an almost quaternion manifold (F, G, H) of first kind (respectively second kind) if the covariant derivatives of F, G, H vanish then (F^c, G^c, H^c) defines a quaternion structure of first kind (respectively second kind) if and only if eqns. (2.10) to (2.18) hold.

Corollary B — If in a manifold having quaternion structure (F, G, H) of first kind (respectively second kind) curl v = curl $\tilde{v} = 0 =$ curl $\check{v}$ then (F^c, G^c, H^c) defines an almost quaternion structure of first kind (respectively second kind) if eqns. (2.10) to (2.18) hold.

§3. The torsion tensor of $F, G \in \mathcal{T}^1_1(M)$ (where $\mathcal{T}^1_1(M)$ is the set of all (1, 1) tensor fields on M) is given by

$$N_{F,G}(X, Y) = [F, G](X, Y) = [FX, GY] - F[GX, Y] - G[X, FY]$$
$$+ [GX, FY] - G[FX, Y] - F[X, GY]$$
$$+ (FG + GF)[X, Y] \qquad \text{...(3.1)}$$

where X, Y are vector fields on M.

It is known (Yano and Patterson 1967)

$$(F^c)^2 = (F^2)^c + (N_{F,F})^v \qquad \text{...(3.2)}$$

where v denotes the vertical lift and

$$F^cG^c + G^cF^c = (FG + GF)^c + (2N_{F,G})^v. \qquad \text{...(3.3)}$$

Thus $(F^c)^2 = -1$ if $N_{F,F} = 0$

$F^cG^c + G^cF^c = 0$ if $N_{F,G} = 0$.

Also $(N_{F,G})^c = N_{F^c,G^c}$ (Yano and Patterson 1967)

$(N_{F,F})^c = N_{F^c,F^c} = 0$ $(N_{F,F} = 0)$.

Thus $(F^c)^2 = -1$ if $N_{F^c,F^c} = 0$.

Similarly $N_{G^c,G^c} = 0$; $N_{H^c,H^c} = 0$.

Further it is known (Yano 1973) that in an almost quaternion manifold M if any two of the six Nijenhuis tensors

$$[F, F], [F, G], [G, H], [H, F], [G, G], [H,H]$$

vanish then the others must vanish. Thus we get :

Theorem 3.1 — In a manifold M with almost quaternion structure (F, G, H) of first kind, (F^c, G^c, H^c) defines a quaternion structure in ${}^cT(M)$ of the same kind if the Nijenhuis tensors $[F^c, F^c]$, $[G^c, G^c]$, $[H^c, H^c]$, $[F^c, G^c]$, $[G^c, H^c]$, $[H^c, F^c]$ vanish.

§4. *Integrability condition for almost quaternion structure* (F^c, G^c, H^c) in ${}^cT(M)$ — Given a quaternion structure (F, G, H) in M, let (F^c, G^c, H^c) define an almost quaternion structure in ${}^cT(M)$. The almost quaternion structure (F^c, G^c, H^c) is integrable if ${}^cT(M)$ can be covered by a system of coordinate neighbourhoods in which the components of F^c, G^c and H^c are all constants (Yano and Ako 1972). In view of (1.3), (1.4), (1.5) this requires that the components of F, G, H are all constants and

$$p_a\partial_{[i}F^a_{h]},\ p_a\partial_{[i}G^a_{h]},\ p_a\partial_{[i}H^a_{h]}$$

are also constants.

Thus (F^c, G^c, H^c) is integrable in ${}^cT(M)$ if (F, G, H) is integrable in M and

$$p_a\partial_{[i}F^a_{h]},\ p_a\partial_{[i}G^a_{h]},\ p_a\partial_{[i}H^a_{h]}$$

are all constants.

Since ${}^cT(M)$ possesses almost quaternion structure, we assume

$$[F^c, F^c] = 0. \qquad \ldots(4.1)$$

Thus the manifold ${}^cT(M)$ is complex analytic and as such is covered by a system of complex coordinate neighbourhoods $\tilde{U}$; $(z^k, \bar{z}^k)$ being the coordinate system in $\tilde{U}$ with respect to which the tensor field F^c of type (1, 1) has components of the form

$$F^c = \begin{pmatrix} iI_n & 0 \\ 0 & -iI_n \end{pmatrix} \qquad \ldots(4.2)$$

where I_n is an $n \times n$ unit matrix

$k, \lambda, \mu \ldots$ vary from $1, 2, \ldots, n$.

$\bar{k}, \bar{\lambda}, \bar{\mu} \ldots$ vary from $n+1, \ldots, 2n$.

We represent the components of the tensor field G^c of type (1, 1) with respect to complex coordinate system by

$$G^c = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}$$

where G_1, G_2, G_3, G_4 are $n \times n$ matrices. Then from

$$F^c G^c + G^c F^c = 0$$

we have

$$G_1 = 0 = G_4.$$

Thus G^c has the form

$$G^c = \begin{pmatrix} 0 & G' \\ G'' & 0 \end{pmatrix} \quad \ldots(4.3)$$

i.e. G^c is hybrid (Yano 1965).

Since $(G^c)^2 = -1$

$$\begin{pmatrix} G'G'' & 0 \\ 0 & G''G' \end{pmatrix} = -1$$

i.e. $G'G'' = G''G' = -I_n$

and from $F^c G^c = H^c$ we see that the components of H^c with respect to this complex coordinate system are

$$H^c = \begin{pmatrix} 0 & iG' \\ -iG'' & 0 \end{pmatrix}. \quad \ldots(4.4)$$

Also

$$[G^c, G^c] = 0. \quad \ldots(4.5)$$

It is proved (Yano and Ako 1972) that under the assumptions (4.1) and (4.5) we can find a symmetric affine connection ∇^c in ${}^cT(M)$ the complete lift of the connection ∇ in M such that

$$\nabla^c F^c = 0; \; \nabla^c G^c = 0; \; \nabla^c H^c = 0.$$

The component $\tilde{\Gamma}^A_{CB}$ of ∇^c in $\pi^{-1}(U)$ where U is a coordinate neighbourhood of M are given by

$$\left.\begin{aligned}
&\tilde\Gamma^h_{ji} = \Gamma^h_{ji};\ \tilde\Gamma^h_{j\bar i} = 0 = \tilde\Gamma^h_{\bar j i} = \tilde\Gamma^h_{\bar j\bar i}\\
&\tilde\Gamma^{\bar h}_{ji} = p_a(\partial_h\Gamma^a_{ji} - \partial_j\Gamma^a_{ih} - \partial_i\Gamma^a_{jh} + 2\Gamma^a_{hb}\Gamma^b_{ji})\\
&\tilde\Gamma^{\bar h}_{j\bar i} = -\Gamma^i_{jh};\ \tilde\Gamma^{\bar h}_{\bar j i} = -\Gamma^j_{hi}\\
&\tilde\Gamma^{\bar h}_{\bar j\bar i} = 0.
\end{aligned}\right\} \quad \ldots(4.6)$$

The components of the curvature tensor of the connection ∇^c are given by :

$$\tilde R^h_{kji} = \partial_k\Gamma^h_{ji} - \partial_j\Gamma^h_{ki} + \Gamma^h_{kt}\Gamma^t_{ji} - \Gamma^h_{jt}\Gamma^t_{ki}$$

$$\tilde R^h_{\bar k ji} = \partial_{\bar k}\Gamma^h_{ji} = 0$$

$$\tilde R^h_{kj\bar i} = 0 = \tilde R^h_{k\bar j i} = \tilde R^h_{k\bar j\bar i}$$

$$\tilde R^h_{\bar k j\bar i} = 0 = \tilde R^h_{\bar k\bar j i} = \tilde R^h_{\bar k\bar j\bar i}$$

$$\begin{aligned}
\tilde R^{\bar h}_{kji} = {} & \partial_k[p_a(\partial_h\Gamma^a_{ji} - \partial_j\Gamma^a_{ih} - \partial_i\Gamma^a_{jh} + 2\Gamma^a_{hb}\Gamma^b_{ji})]\\
& - \partial_j[p_a(\partial_h\Gamma^a_{ki} - \partial_k\Gamma^a_{ih} - \partial_i\Gamma^a_{kh} + 2\Gamma^a_{hb}\Gamma^b_{ki})]\\
& + \Gamma^t_{jh}[p_a(\partial_t\Gamma^a_{ki} - \partial_k\Gamma^a_{it} - \partial_i\Gamma^a_{kt} + 2\Gamma^a_{tb}\Gamma^b_{ki})]\\
& + p_a(\partial_h\Gamma^a_{kt} - \partial_k\Gamma^a_{th} - \partial_t\Gamma^a_{kh} + 2\Gamma^a_{hb}\Gamma^b_{kt})\Gamma^t_{ji}\\
& - \Gamma^t_{kh}[p_a(\partial_t\Gamma^a_{ji} - \partial_j\Gamma^a_{it} - \partial_i\Gamma^a_{jt} + 2\Gamma^a_{tb}\Gamma^b_{ji})]\\
& - p_a(\partial_h\Gamma^a_{jt} - \partial_j\Gamma^a_{th} - \partial_t\Gamma^a_{jh} + 2\Gamma^a_{hb}\Gamma^b_{jt})\Gamma^t_{ki}
\end{aligned}$$

$$\tilde R^{\bar h}_{\bar k ji} = \partial_j\Gamma^k_{hi} - \Gamma^k_{ht}\Gamma^t_{ij} - \Gamma^i_{jh}\Gamma^k_{ti}$$

$$R^{\bar{h}}_{kj\bar{i}} = -\partial_k \Gamma^{j}_{hi} + \Gamma^{j}_{ht}\Gamma^{t}_{ki} + \Gamma^{t}_{kh}\Gamma^{j}_{ti}$$

$$\tilde{R}^{h}_{kj\bar{i}} = -\partial_k \Gamma^{i}_{jh} + \partial_j \Gamma^{i}_{kh} + \Gamma^{t}_{kh}\Gamma^{i}_{jt} - \Gamma^{t}_{jh}\Gamma^{i}_{kt}$$

$$\tilde{R}^{\bar{h}}_{\bar{k}j\bar{i}} = -\partial_{\bar{k}}\Gamma^{i}_{jh} = 0$$

$$\tilde{R}^{\bar{h}}_{\bar{k}\bar{j}i} = 0$$

$$\tilde{R}^{h}_{\bar{k}\bar{j}\bar{i}} = 0 = \tilde{R}^{\bar{h}}_{\bar{k}\bar{j}\bar{i}}.$$

Since

$$\nabla^c F^c = 0$$

i.e. $$\partial_C \tilde{F}^A_B + \tilde{\Gamma}^A_{CD}\tilde{F}^D_B - \tilde{\Gamma}^D_{CB}\tilde{F}^A_D = 0.$$

In view of the components of F^c and ∇^c the above equation can be expressed in six different ways as :

$$\left.\begin{aligned}
&\partial_j F^h_i + \Gamma^h_{jt} F^t_i - \Gamma^t_{ji} F^h_t = 0 \\
&\partial_j (p_a \partial_{[i} F^a_{h]}) - p_a(\partial_t \Gamma^a_{ji} - \partial_j \Gamma^a_{it} - \partial_i \Gamma^a_{jt} + 2\Gamma^a_{tb}\Gamma^b_{ji}) F^t_h \\
&\quad - \Gamma^t_{ji} p_a \partial_{[t} F^a_{h]} - \Gamma^t_{jh} p_a \partial_{[i} F^a_{t]} + p_a(\partial_h \Gamma^a_{jt} - \partial_j \Gamma^a_{th} \\
&\qquad - \partial_t \Gamma^a_{jh} + 2\Gamma^a_{hb}\Gamma^b_{jt}) F^t_i = 0 \\
&\qquad - \Gamma^j_{ht} F^t_i + \Gamma^j_{ti} F^t_h = 0 \\
&\partial_j F^i_h - \Gamma^t_{jh} F^i_t + \Gamma^i_{jt} F^t_h = 0 \\
&\partial_{\bar{j}} F^i_h = 0 = \partial_{\bar{j}} F^h_i .
\end{aligned}\right\} \quad \ldots(4.7)$$

Similarly

$$\left.\begin{aligned}
&\partial_j G^h_i + \Gamma^h_{jt} G^t_i - \Gamma^t_{ji} G^h_t = 0 \\
&\partial_j(p_a \partial_{[i} G^a_{h]}) + p_a(\partial_h \Gamma^a_{jt} - \partial_j \Gamma^a_{th} - \partial_t \Gamma^a_{jh} + 2\Gamma^a_{hb} \Gamma^b_{jt}) G^t_i \\
&\quad - \Gamma^t_{ji}\, p_a \partial_{[t} G^a_{h]} - \Gamma^t_{jh}\, p_a \partial_{[i} G^a_{t]} - p_a(\partial_t \Gamma^a_{ji} - \partial_j \Gamma^a_{it} \\
&\qquad - \partial_i \Gamma^a_{jt} + 2\Gamma^a_{tb} \Gamma^b_{ji}) G^t_h = 0 \\
&\qquad - \Gamma^j_{ht} G^t_i + \Gamma^j_{ti} G^t_h = 0 \\
&\partial_j G^i_h - \Gamma^t_{jh} G^i_t + \Gamma^i_{jt} G^t_h = 0 \\
&\partial_{\bar{j}} G^i_h = 0 = \partial_{\bar{j}} G^h_i
\end{aligned}\right\} \qquad \ldots(4.8)$$

and

$$\left.\begin{aligned}
&\partial_j H^h_i + \Gamma^h_{jt} H^t_i - \Gamma^t_{ji} H^h_t = 0 \\
&\partial_j(p_a \partial_{[i} H^a_{h]}) + p_a(\partial_h \Gamma^a_{jt} - \partial_j \Gamma^a_{th} - \partial_t \Gamma^a_{jh} + 2\Gamma^a_{hb} \Gamma^b_{jt}) H^t_i \\
&\quad - \Gamma^t_{ji}\, p_a \partial_{[t} H^a_{h]} - \Gamma^t_{jh}\, p_a \partial_{[i} H^a_{t]} \\
&- p_a(\partial_t \Gamma^a_{ji} - \partial_j \Gamma^a_{it} - \partial_i \Gamma^a_{jt} + 2\Gamma^a_{tb} \Gamma^b_{ji}) H^t_h = 0 \\
&\quad - \Gamma^j_{ht} H^t_i + \Gamma^j_{ti} H^t_h = 0 \\
&\partial_j H^i_h - \Gamma^t_{jh} H^i_t + \Gamma^i_{jt} H^t_h = 0 \\
&\partial_{\bar{j}} H^i_h = 0 = \partial_{\bar{j}} H^h_i = 0
\end{aligned}\right\} \qquad \ldots(4.9)$$

Thus from (4.7), (4.8), (4.9) it follows that (F^c, G^c, H^c) is integrable if and only if Γ^h_{ji} vanish.

This leads to that all the components of $\tilde{R}$ vanish.

Hence we obtain :

Theorem 4.1 — A necessary and sufficient condition that an almost quaternion structure (F^c, G^c, H^c) is integrable is

$$[F^c, F^c] = 0; [G^c, G^c] = 0 \text{ and } \tilde{R} = 0$$

where $\tilde{R}$ is the curvature tensor of a symmetric affine connection ∇^c in ${}^cT(M)$ s.t. $\nabla^c F^c = 0$; $\nabla^c G^c = 0$, $\nabla^c H^c = 0$, F^c, G^c, H^c has components given by (4.2), (4.3) and (4.4).

§5. Let us now consider horizontal lifts. It is known (Yano and Patterson 1967a) that the horizontal lift of a tensor field F of type (1, 1) in M is also a tensor field of the same type in ${}^cT(M)$ denoted by F^L, having the following components.

$$\tilde{F}_i^h = F_i^h; \quad \tilde{F}_{\bar{i}}^h = 0$$

$$\tilde{F}_i^{\bar{h}} = -\Gamma_{ia}F_h^a + \Gamma_{ha}F_i^a; \quad F_{\bar{i}}^{\bar{h}} = F_h^i \qquad \ldots(5.1)$$

where $\Gamma_{ji} = p_a\Gamma_{ji}^a$; Γ_{ji}^a being the components of the symmetric affine connection in M.

Thus

$$\tilde{F}_B^A = \begin{pmatrix} F_i^h & 0 \\ -\Gamma_{ia}F_h^a + \Gamma_{ha}F_i^a & F_h^i \end{pmatrix}. \qquad \ldots(5.2)$$

It is known (Yano and Patterson 1967a) that

$(F^L)^2 = (F^2)^L$. Therefore $(F^L)^2 = -1$ since $F^2 = -1$. Similarly $(G^L)^2 = -1 = (H^L)^2$.

Again $\quad F^L G^L = \tilde{F}_B^A \tilde{G}_C^B$

$$= \begin{pmatrix} F_j^i & 0 \\ -\Gamma_{ja}F_i^a + \Gamma_{ia}F_j^a & F_i^j \end{pmatrix} \begin{pmatrix} G_k^j & 0 \\ -\Gamma_{ka}G_j^a + \Gamma_{ja}G_k^a & G_k^j \end{pmatrix}$$

$$= \begin{pmatrix} H_k^i & 0 \\ N_{ik} & H_i^k \end{pmatrix}$$

where

$$N_{ik} = (-\Gamma_{ja}F_i^a + \Gamma_{ia}F_j^a)G_k^j$$

$$+ F_i^j(-\Gamma_{ka}G_j^a + \Gamma_{ja}G_k^a).$$

Thus $F^L G^L = H^L$ if

$$\begin{pmatrix} H_k^i & 0 \\ -\Gamma_{ka} H_i^a + \Gamma_{ia} H_k^a & H_i^k \end{pmatrix} \text{coincides with}$$

$$\begin{pmatrix} H_k^i & 0 \\ N_{ik} & H_i^k \end{pmatrix}$$

i.e. if

$$N_{ik} = -\Gamma_{ka} H_i^a + \Gamma_{ia} H_k^a .$$

In view of the relations (1.1) the above equation is identically true, i.e. $F^L G^L = H^L$.

Similarly $G^L H^L = F^L$ and $H^L F^L = G^L$.

Also it can be easily checked that $G^L F^L = -H^L$, $H^L G^L = -F^L$ and $F^L H^L = -G^L$.

We thus conclude :

Theorem 5.1 — In an almost quaternion manifold M with almost quaternion structure (F, G, H) of first kind (respectively second kind); (F^L, G^L, H^L) is an almost quaternion structure of first kind (respectively second kind) in ${}^cT(M)$.

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