# SLOW UNSTEADY FLOW IN AXISYMMETRIC TUBE OF VARYING RADIUS

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The slow unsteady flow of a viscous incompressible fluid through an axisymmetric tube of variable radius is considered under Stokes' approximations. Series solution in powers of the small ratio of mean radius to a characteristic length along the axis has been obtained for the stream function and pressure. But no restrictions are required to be placed on the ratio of maximum constriction to mean radius or on that of the unsteady to steady mean flux across the cross-sections of the tube. Shear stress on the wall has also been calculated. A particular case of radius varying sinusoidally is discussed.

## 1. INTRODUCTION

The flow in tubes of varying radius has been studied with a view to understanding the flow in arteriosclerotic blood vessels and in pipes which have developed roughness on their walls. Manton (1971) considered steady flow in a tube of slowly varying radius. Chow and Soda (1972) studied steady flow in a tube having an arbitrary form of roughness on its wall, the spread of roughness being large in comparison to the mean radius of the tube. The slow unsteady flow caused by a pulsating pressure gradient in a circular cylindrical tube having small roughness on its wall was studied by Verma and Gaur (1970). They used Fourier sine transform to eliminate the axial variable. But, as pointed out by Bhatnagar and Mohan Rao (1965), this technique introduces implicitly the assumption that the perturbations in the velocity vanish at infinity even when the radius varies continuously as a periodic function of the axial variable. Ramchandra Rao and Devanathan (1973) considered slow pulsatile flow in tubes of slowly varying cross-section by expanding the stream function in powers of the supposedly small ratio of the orders of magnitude of typical oscillatory and steady axial velocities.

This paper deals with the slow unsteady flow in an axisymmetric tube of arbitrarily varying radius. The flow is viewed as being composed of a pulsatile part superposed on a steady part, but no restriction is placed on the ratio of the mass flux caused by the two parts. The magnitude of constriction may not be small in comparison to the mean radius of the tube. Instead, the mean radius is assumed to be small in comparison to a characteristic length along the axis of the tube. A particular case of flow in a tube having sinusoidal boundary is discussed with the help of numerical results.

## 2. Equations of Motion

Cylindrical polar coordinates $(\bar{x}, \bar{r}, \theta)$ are employed, such that $\bar{r} = 0$ is the axis and $\bar{r} = \bar{R}(\bar{x})$ is the boundary of the tube. The variable of time is denoted by $\bar{t}$. The density $\rho$ and the kinematic viscosity $\nu$ of the fluid are constant. The velocity components in the axial and radial directions are $\bar{u}$ and $\bar{v}$, such that

$$\bar{u} = (\partial\bar{\psi}/\partial\bar{r})/\bar{r}, \quad \bar{v} = -(\partial\bar{\psi}/\partial\bar{x})/\bar{r}. \qquad \text{...(2.1)}$$

The vorticity $\Omega$ is equal to

$$\partial\bar{u}/\partial\bar{r} - \partial\bar{v}/\partial\bar{x}. \qquad \text{...(2.2)}$$

It satisfies the following equation obtained by eliminating the fluid pressure $\bar{p}$ from the equations of axisymmetric motion, in which the inertia terms have been neglected under Stokes' approximation:

$$\partial\Omega/\partial\bar{t} = \nu\,[\partial^2\Omega/\partial\bar{x}^2 + \partial(\{\partial(\bar{r}\Omega)/\partial\bar{r}\}/\bar{r})/\partial\bar{r}]. \qquad \text{...(2.3)}$$

The condition of 'no slip' at the tube wall requires that

$$\partial\bar{\psi}/\partial\bar{r} = 0 = \partial\bar{\psi}/\partial\bar{x} \quad \text{at} \quad \bar{r} = \bar{R}(\bar{x}). \qquad \text{...(2.4a)}$$

The condition of symmetry about the axis implies that

$$(\partial\bar{\psi}/\partial\bar{x})/\bar{r} = 0 = \partial\,[(\partial\bar{\psi}/\partial\bar{r})/\bar{r}]\partial\bar{r} \quad \text{at} \quad \bar{r} = 0. \qquad \text{...(2.4b)}$$

Since the mass flux at any instant is same across all cross-sections of the tube,

$$\int_0^{\bar{R}} 2\pi\bar{r}\,[(\partial\bar{\psi}/\partial\bar{r})/\bar{r}]\,d\bar{r} = \bar{Q}(\bar{t}).$$

On choosing

$$\bar{\psi} = 0 \quad \text{at} \quad \bar{r} = 0, \qquad \text{...(2.4c)}$$

this reduces to

$$\bar{\psi}(\bar{t}) = \bar{Q}(\bar{t})/(2\pi) \quad \text{at} \quad \bar{r} = \bar{R}. \qquad \text{...(2.4d)}$$

The equations of motion are linear in the absence of inertia terms, so that the flow may be supposed to be made up of a pulsatile part superposed on a steady part, such that

$$\bar{Q}(\bar{t}) = Q^*\,[1 + k\exp(i\sigma\bar{t})]. \qquad \text{...(2.5)}$$

On the same pattern, let

$$\left.\begin{aligned} \bar{\psi}(\bar{x},\bar{r},\bar{t}) &= \bar{\psi}^*(\bar{x},\bar{r}) + k\tilde{\psi}(\bar{x},\bar{r})\exp(i\sigma\bar{t}), \\ \Omega(\bar{x},\bar{r},\bar{t}) &= \bar{\omega}^*(\bar{x},\bar{r}) + k\tilde{\omega}(\bar{x},\bar{r})\exp(i\sigma\bar{t}), \\ \bar{p}(\bar{x},\bar{r},\bar{t}) &= \bar{p}^*(\bar{x},\bar{r}) + k\tilde{p}(\bar{x},\bar{r})\exp(i\sigma\bar{t}). \end{aligned}\right\} \qquad \text{...(2.6)}$$

Only real parts of the complex quantities introduced above as well as of those arising in the sequel are physically significant.

Now let $\lambda$ be a characteristic length along the axis of the tube over which significant changes in the flow variables take place, and let $r_0$ be the mean radius of the tube. The characteristic velocity is defined by the relation

$$\pi r_0^2 u_0 = Q^*.$$

The following dimensionless variables are introduced:

$$r = \bar{r}/r_0, \quad x = \bar{x}/\lambda, \quad t = \bar{t}\sigma,$$

$$u = \bar{u}/u_0, \quad v = \bar{v}/u_0.$$

Both the starred and unstarred variables for the stream function, vorticity and pressure are non-dimensionalized by the relations

$$\psi = \bar{\psi}/(u_0 r_0^2), \quad \omega = \bar{\omega} r_0/u_0, \quad p = \bar{p}\delta Re/(\rho u_0^2),$$

where $Re = u_0 r_0/\nu$ and $\delta = r_0/\lambda$ are small.

Also, $s = \sigma r_0^2/\nu$ appears as a dimensionless parameter, depending upon the frequency of oscillation of the unsteady part. The boundary of the tube is now given by $r = R(x)$.

The functions $\psi^*$ and $\psi$ separately satisfy the boundary conditions of the form (2.4*a–c*), but the condition (2.4*d*) is now simplified to

$$\psi^* = \psi = 1/2 \quad \text{at} \quad r = R(x). \qquad \ldots(2.7)$$

## 3. Solution of the Problem

The equations are solved by assuming series expansions for the vorticity and the stream function in ascending powers of $\delta$. The steady part is found to be given by

$$\psi^* = \psi^*_0 + \delta^2 \psi^*_2 + 0(\delta^4), \qquad \ldots(3.1)$$

where

$$\psi^*_0 = (r^2/R^2)[1 - r^2/(2R^2)], \qquad \ldots(3.2)$$

$$\psi^*_2 = [r^2(1 - r^2/R^2)^2/(6R^2)] \times [5(\partial R/\partial x)^2 - R\partial^2 R/\partial x^2]. \qquad \ldots(3.3)$$

Similarly, the unsteady part is found to be given by

$$\psi = \psi_0 + \delta^2 \psi_2 + 0(\delta^4), \qquad \ldots(3.4)$$

where

$$\psi_0 = z[2J_1(z) - zJ_0]/[2Z^2 J_2], \qquad \ldots(3.5)$$

$$\psi_2 = izJ_1 (\partial^2 Z/\partial x^2) [Z^2 J_3 \{zJ_0 - 2J_1 (z)\} + zJ_2 \{4ZJ_2 (z) - z^2 J_1\}]/[8\, sZ^3 J_2^3] + iz (\partial Z/\partial x)^2 [zf_1 (Z) + z^3 f_2 (Z) + J_1 (z) f_3 (Z) + zJ_0 (z) f_4 (Z)]/[8\, sZ^5 J_2^4], \quad \ldots(3.6)$$

$$f_1 (Z) = Z^3 J_0 J_3 [ZJ_0 J_2 + J_1 J_2 - 2ZJ_1^2] - Z^4 J_1 J_2^2 [J_0 + 2J_2],$$

$$f_2 (Z) = J_1 J_2 [2Z^2 J_0^2 - 5Z J_0 J_1 + 2 (Z^2 + 1) J_1^2],$$

$$f_3 (Z) = 2Z^3 J_1^2 [3J_2 - 2ZJ_1],$$

$$f_4 (Z) = 4ZJ_2 [Z^2 J_0^2 - ZJ_0 J_1 - 2 (1 - Z^2) J_1^2]. \quad \ldots(3.7)$$

The argument of the Bessel functions, where not specified in these or the following expressions, is understood to be $Z$,

$$z = r\sqrt{(i^3 s)}, \quad Z = R\sqrt{(i^3 s)}.$$

On evaluating the velocity components, the radial component is found to be caused entirely by the variation of the tube radius, and to be one order higher in the small parameter $\delta$ than the axial component. By substituting these components in the equations of motion, the steady and unsteady parts of the hydrodynamic pressure are found to be given up to third order terms in $\delta$ by

$$p^* = \text{const.} - (8/3) \int_0^x [\{3 + 4\delta^2 (\partial R/\partial x)^2\}/R^4]\, dx + [4\delta^2 (1 - 6r^2/R^2)/(3R^3)] (\partial R/\partial x). \quad \ldots(3.8)$$

$$p = \text{const.} + s^2 \int_0^x [J_0/(Z^2 J_2)]\, dx - (is\delta^2/2) \int_0^x [(\partial Z/\partial x)^2 J_1^2 \{J_1 J_3 - 2J_2^2\}/(Z^2 J_2^4)]\, dx + is\delta^2 (\partial Z/\partial x) J_1 [2z^2 J_1 J_2 + ZJ_3 \{ZJ_2 - 2J_1\}]/(4Z^2 J_2^3). \quad \ldots(3.9)$$

The shear stress $\bar{\tau}$ at the boundary is equal to $\rho\nu\Omega$. Therefore, if

$$\bar{\tau} = \bar{\tau}^* + k\bar{\tau} \exp (i\sigma\bar{t})$$

and

$$\tau^* = \bar{\tau}^* Re/(\rho u_0^2), \quad \tau = \bar{\tau}\, Re/(\rho u_0^2),$$

then, to the same order terms in $\delta$,

$$\tau^* = -(4/R^3) [1 - (\delta^2/3) \{2 (\partial R/\partial x)^2 - R (\partial^2 R/\partial x^2)\}], \quad \ldots(3.10)$$

$$\tau = -(is^3)^{1/2} J_1/[Z^2 J_2] - \delta^2 (i^3 s)^{1/2} (\partial^2 Z/\partial x^2) J_1 [J_1^2 - 2J_0 J_2]/[4ZJ_2^3] + \delta^2 (i^3 s)^{1/2} (\partial Z/\partial x)^2 [J_1^3 \{2ZJ_1 - 3J_2\} + 2J_0 J_2 \{ZJ_0 J_2 + J_1 J_2 - 2ZJ_1^2\}]/[4Z^2 zJ_2^4]. \quad \ldots(3.11)$$

## 4. Discussion

The results are presented graphically for a tube having sinusoidal boundary given by

$$\bar{r} = r_0 [1 + \varepsilon \sin (2\pi\bar{x}/\lambda)].$$

$\varepsilon$ has been taken equal to 0·2, although larger values are permissible; $\delta$ has been taken equal to 0·25, but smaller values would ensure better convergence. The parameter $s$ has been taken as unity. Fig. 1 shows profiles of the unsteady part of the axial velocity $u$ drawn against $r$ at four typical sections of the tube for the instants $t = 0$ and $t = \pi/4$. Fig. 2 similarly shows the profiles of the unsteady radial velocity $v$. Contrary to the inference drawn by Verma and Gaur (1970), the profiles of $v$ are seen to be symmetrical about the axis instead of having any points of inflexion on it. The radial velocity is zero at all points of the maximum or the minimum cross-section of the tube at all times. Fig. 3 shows the phase difference between the axial velocity and the exciting pulse. The phase difference would be larger for larger values of $s$. The profiles of the steady part of velocity are same as those of the unsteady part at $t = 0$.

Curves showing skin friction due to the unsteady part of the flow as a function of the axial coordinate have been drawn in Fig. 4 for the instants $t = 0$ and $t = \pi/4$. The maximum skin friction is at points of maximum constriction. The skin friction due to the steady part of the flow is also very nearly equal to

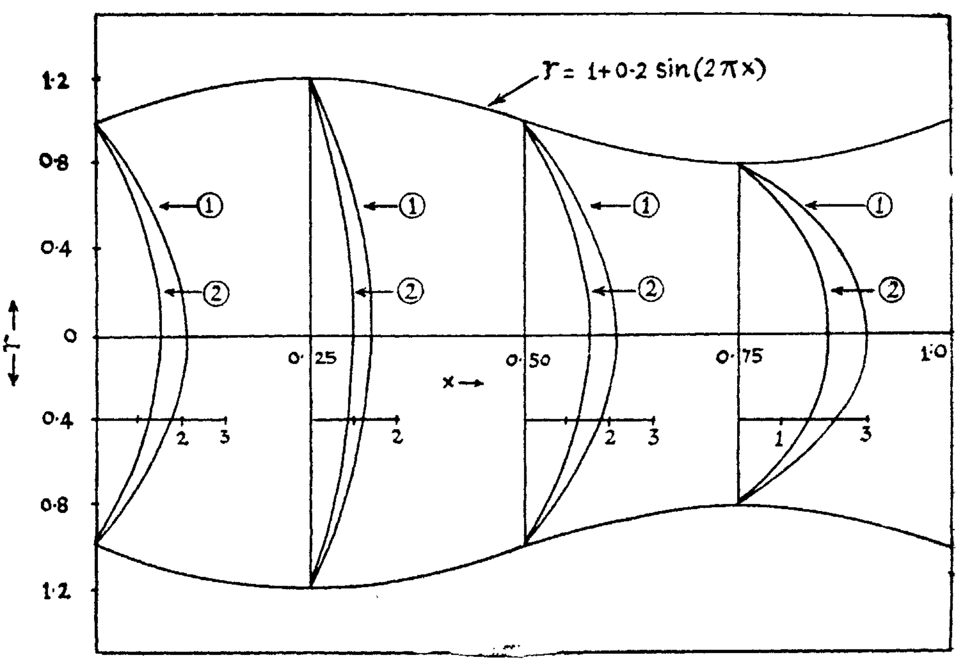


Fig. 1. Unsteady axial velocity profiles at different sections [(1) at $t = 0$, (2) at $t = \pi/4$].

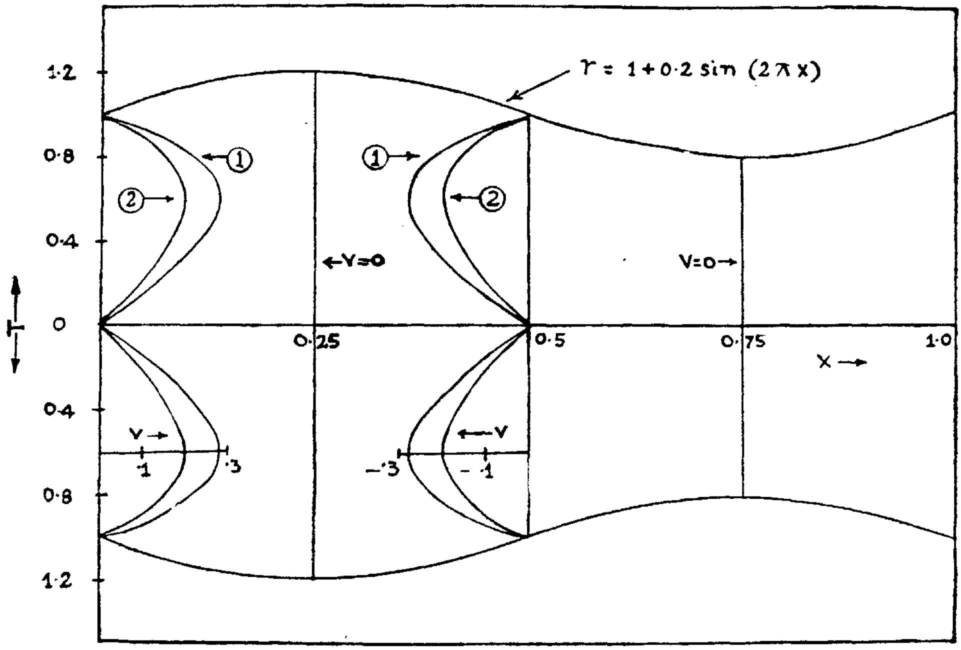


FIG. 2. Unsteady radial velocity profiles at different sections [(1) at $t = 0$, (2) at $t = \pi/4$].

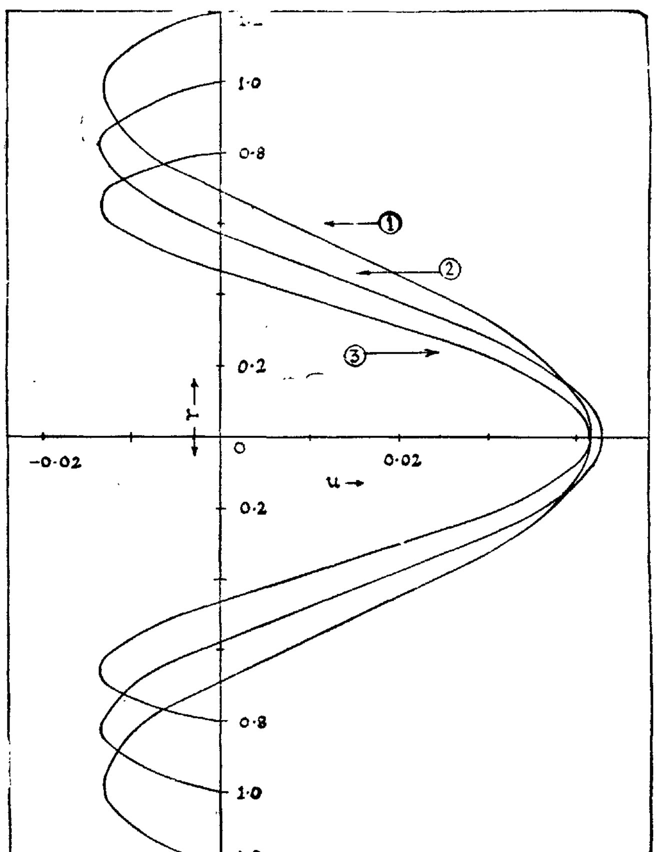


FIG 3. Unsteady axial velocity profiles at $t = \pi/2$ [(1) at $x = 0{\cdot}25$, (2) at $x = 0$ and $x = 0{\cdot}5$, (3) at $x = 0{\cdot}75$].

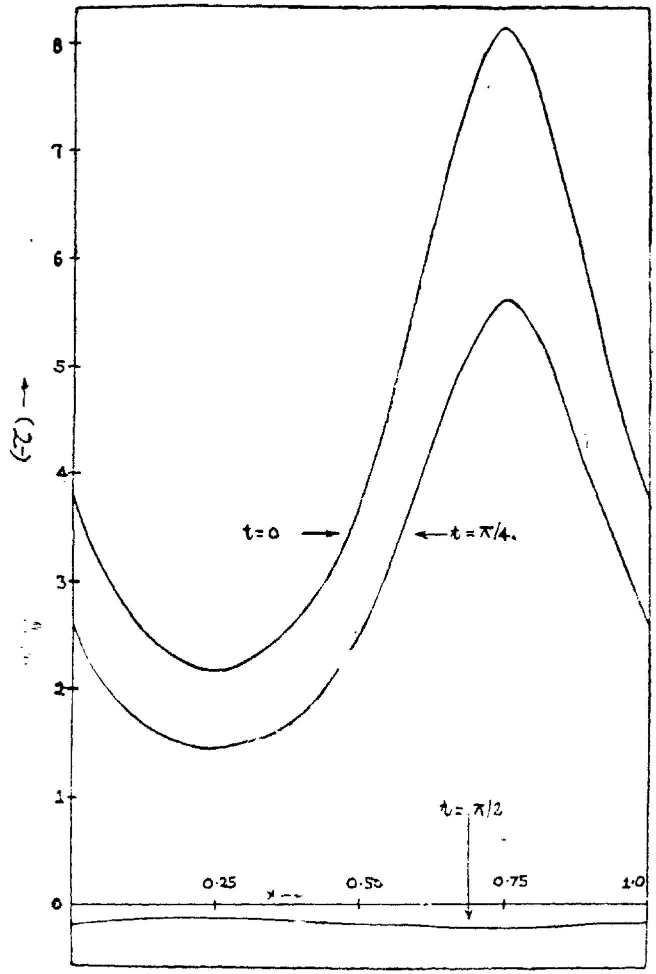


FIG. 4. Unsteady shear stress against axial variable.

that due to the unsteady part at $t = 0$. Consequently, if $k$ be equal to unity, then at $t = \pi$, the combined shear stress vanishes at all points of the boundary. Obviously, this would be indicative of a momentary pause in the flow and not of any separation. The solution presented here is symmetrical about the planes of maximum as well as of minimum cross-sections and is unable to predict separation or the secondary flow induced by the oscillatory part on account of the inertia terms having been neglected.

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