HARMONIC SUMMABILITY OF FOURIER SERIES

by B. L. GUPTA *and* L. P. GAUTAM, *Department of Mathematics, Government Engineering College, Rewa*

(*Received* 8 *September* 1975)

Hille and Tamarkin (1932) have proved the harmonic summability of Fourier series. In the present paper, we have generalized the theorem of Hille and Tamarkin.

§ 1. A series $\sum_{n=0}^{\infty} U_n$ with partial sums $S_n = \sum_{k=0}^{n} U_k$ is said to be summable by Nörlund method defined by the sequence $\{p_n\}$ or simply summable (N, p_n), if t_n tends to a limit as $n \to \infty$, where

$$t_n = (P_n)^{-1} \sum_{k=0}^{n} p_{n-k} S_k. \qquad \text{...(1·1)}$$

If we choose

$$p_n = \frac{1}{(n+1)} \qquad \text{...(1·2)}$$

and consequently

$$P_n = \sum_{k=0}^{n} \frac{1}{k+1} \sim \log n \qquad \text{...(1·3)}$$

the transform t_n reduces to

$$\frac{1}{\log n} \sum_{k=0}^{n} \frac{S_{n-k}}{k+1}. \qquad \text{...(1·4)}$$

This method of summability is known as the Harmonic method of summability.

Let $f(x)$ be a function of x, integrable in the sense of Lebesque over the interval $(-\pi, \pi)$ and periodic with period 2π outside the interval. Let its Fourier series be

$$f(x) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \qquad \text{...(1·5)}$$

We write

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\} \qquad \ldots(1\cdot6)$$

and

$$P_M(t) = \sum_{k=M+1}^{\infty} \frac{\sin(M-k+\frac{1}{2})\,t}{k+1} \qquad \ldots(1\cdot7)$$

§ 2. The following theorem on harmonic summability of Fourier series is due to Hille and Tamarkin (1932).

Theorem—If

$$\text{(i)} \int_0^t |\phi(u)|\,du = 0(t) \qquad \ldots(2\cdot1)$$

and

$$\text{(ii)} \lim_{n\to\infty} \frac{1}{\log n} \int_{\pi/n}^{n} \frac{|\phi(t+x) - \phi(t)|}{t} \log\frac{1}{t}\,dt = 0 \qquad \ldots(2\cdot2)$$

Then the Fourier series is summable $(H, 1)$ to the sum zero at the point $t = x$, where $x = \pi/n$.

In this paper, we generalize the above theorem by proving the following:

Theorem A—If

$$\Phi(t) = \int_0^t |\phi(u)|\,du = 0(t), \quad \text{as } t \to 0 \qquad \ldots(2\cdot3)$$

and further if for any n there is an $m = m(n) > n$ such that

$$\int_{1/m}^{1/(m-n)} \frac{|\phi(t) - \phi(t-\pi/m)|}{t} \log(1/t)\,dt = 0(\log n) \text{ as } n \to \infty \qquad \ldots(2\cdot4)$$

and

$$\sum_{\nu=n}^{m} (|a_\nu| + |b_\nu|) \to 0, \quad \text{as} \quad n \to \infty, \qquad \ldots(2\cdot5)$$

Then the Fourier series $(1\cdot5)$ is summable $(H, 1)$ at the point $t = x$ where a_ν and b_ν are the Fourier coefficients of f.

If $(m-n)$ is bounded, then the above theorem reduces to the theorem of Hille and Tamarkin.

§ 3. To prove the theorem, we need the following estimates:

$$\sum_{k=0}^{M} \frac{\sin(M-k+\frac{1}{2})\,t}{(k+1)\sin\frac{1}{2}t} = O(M\log M). \qquad \ldots(3\cdot1)$$

$$\sum_{k=M+1}^{\infty} \frac{\sin(M-k+\frac{1}{2})t}{k+1} = O\left(\frac{1}{Mt}\right). \qquad \ldots(3\cdot 2)$$

PROOF: The estimate (3·1) can be obtained from the relation

$$|\sin(M-k+\tfrac{1}{2})t| \leqslant (M+\tfrac{1}{2})t \quad \text{for} \quad k \leqslant M.$$

To prove (3·2), we observe that

$\sum_{k=M+1}^{\infty} \frac{i(M-k+\frac{1}{2})t}{k+1}$ converges for $t \neq 0$ and its sum does not exceed

$\frac{4}{(M+2)\,|1-e^{-it}|}$ in absolute value.

Further, we set

$$\sum_{k=0}^{M} \frac{\sin(M-k+\frac{1}{2})t}{k+1}$$

$$= \sum_{k=0}^{\infty} \frac{\sin(M-k+\frac{1}{2})t}{k+1} - \sum_{k=M+1}^{\infty} \frac{\sin(M-k+\frac{1}{2})t}{k+1}$$

$$= I\left[\sum_{k=0}^{\infty} \frac{e^{i(M-k+1/2)t}}{k+1}\right] - \sum_{k=M+1}^{\infty} \frac{\sin(M-k+\frac{1}{2})t}{k+1}$$

$$= I\left[e^{i(M+3/2)t} \sum_{k=0}^{\infty} \frac{e^{-i(k+1)t}}{k+1}\right] - R_M(t)$$

$$= I[-e^{-i(M+3/2)t} \log(1-e^{-it})] - R_M(t)$$

$$= I[-e^{i(M+3/2)t} \log(2\sin^2\tfrac{1}{2}t + 2i\sin\tfrac{1}{2}t\cos\tfrac{1}{2}t)] - R_M(t)$$

$$= \sin\left(M+\frac{3}{2}\right)t \log\frac{1}{2\sin\frac{1}{2}t} + \left(\frac{1}{2}\pi - \frac{1}{2}t\right)\cos\left(M+\frac{3}{2}\right)t - R_M(t). \qquad \ldots(3\cdot 3)$$

Let

$$N_{n+1}(t) = -\tfrac{1}{2} + \sum_{\nu=0}^{n+1} e^{i\nu t}$$

$$= \frac{\sin(n+\frac{3}{2})t}{2\sin\frac{1}{2}t} - i\,\frac{\cos(n+\frac{3}{2}t)}{2\sin\frac{1}{2}t} + \frac{i\sin t}{4\sin^2\frac{1}{2}t},$$

and $\{\lambda_n\}$ be a sequence of positive number which will be determined later and we write

$$\sum_{n=N}^{M} \lambda_n N_n(t) = \sum_{n=N}^{M} \lambda_n [N_{M+1}(t) - \sum_{m=n+1}^{M+n} e^{imt}]$$

$$= (\Lambda_M - \Lambda_N) N_{M+1}(t) - \sum_{n=N+1}^{M+1} (\sum_{m=n}^{n-1} \lambda_m) e^{int}$$

$$N_{M+1}(t) = \frac{1}{\Lambda_M - \Lambda_N} \sum_{n=N}^{M} \lambda_n N_n(t) + \frac{1}{\Lambda_M - \Lambda_N} \sum_{n=N+1}^{M+1} (\Lambda_{n-1} - \Lambda_{N-1}) e^{int}, \quad \ldots(3.4)$$

Equating the real and imaginary part, we get

$$\frac{\sin (M + \frac{3}{2}) t}{2 \sin \frac{1}{2} t} = \frac{1}{\Lambda_M - \Lambda_N} \sum_{n=N}^{M} \lambda_n \frac{\sin (n + \frac{1}{2}) t}{\sin \frac{1}{2} t}$$

$$+ \frac{1}{\Lambda_M - \Lambda_N} \sum_{n=N+1}^{M+1} (\Lambda_{n-1} - \Lambda_{N-1}) \cos nt \quad \ldots(3.5)$$

$$\frac{\cos (M + \frac{3}{2}) t}{2 \sin \frac{1}{2} t} = \frac{1}{\Lambda_M - \Lambda_N} \sum_{n=N}^{M} \lambda_n \frac{\cos (n + \frac{1}{2}) t}{\sin \frac{1}{2} t}$$

$$+ \frac{1}{\Lambda_M - \Lambda_N} \sum_{n=N+1}^{M+1} (\Lambda_{n-1} - \Lambda_{N-1}) \sin nt \quad \ldots(3.6)$$

where

$$\Lambda_n = \sum_{\nu=1}^{n} \lambda_\nu.$$

We write

$$\mu = [\tfrac{1}{2}(M + N)] \qquad \text{and} \qquad \nu = [\tfrac{1}{2}(M - N)]$$

and suppose that $\lambda_n = 0$ for n outside the internal $(\mu - \nu, \mu + \nu)$ and $\lambda_{\mu+\nu} = \lambda_{\mu-\nu}$ for $0 < n \leqslant \nu$, we have

$$\sum_{n=N}^{M} \lambda_n e^{i(n+1/2)t}$$

$$= e^{(\mu+1/2)it} \sum_{n=-\nu}^{\nu} \lambda_{\mu+n} e^{int}$$

$$= 2e^{(\mu+1/2)it} \left(\frac{\lambda\mu}{2} + \sum_{n=1}^{\nu} \lambda_{\mu+n} \cos nt\right).$$

Therefore,

$$\sum_{n=N}^{M} \lambda_n \cos (n + \tfrac{1}{2}) t = \cos (\mu + \tfrac{1}{2}) t \left(\frac{\lambda\mu}{2} + \sum_{n=1}^{\nu} \lambda_{\mu+n} \cos nt\right) \quad \ldots(3\cdot 7)$$

$$\sum_{n=N}^{M} \lambda_n \sin (n + \tfrac{1}{2}) t = \sin (\mu + \tfrac{1}{2}) t \left(\frac{\lambda\mu}{2} + \sum_{n=1}^{\nu} \lambda_{\mu+n} \cos nt\right). \quad \ldots(3\cdot 8)$$

Let $0 < \delta_N < \pi$ and $h(t)$ be the characteristic function of the interval $(-\delta_N, \delta_N)$ with period 2π and we take $\lambda\mu/2 + \sum_{n=1}^{\nu} \lambda_{\mu+n} \cos nt$ as the νth Cesáro mean of Fourier series of $h(t)$, i.e.,

$$\frac{\lambda\mu}{2} + \sum_{n=1}^{\nu} \lambda_{\mu+n} \cos nt = \frac{1}{\pi(\nu+1)} \int_{-\delta_N}^{\delta_N} K_\nu(t-u)\, du$$

$$= \frac{1}{\pi(\nu+1)} \int_{t-\delta_N}^{t+\delta_N} K_\nu(u)\, du$$

$$= \delta_N + \sum_{n=1}^{\nu} \left(1 - \frac{n}{\nu+1}\right) \int_{-\delta_N}^{\delta_N} \cos n(t-u)\, du$$

$$= \delta_N + \sum_{n=1}^{\nu} \left(1 - \frac{n}{\nu+1}\right) \frac{1}{n} [\sin n(t+\delta_N) - \sin n(t-\delta_N)]$$

$$= \delta_N + \sum_{n=1}^{\nu} \left(1 - \frac{n}{\nu+1}\right) \frac{2 \sin n\delta_N}{n} \cos nt.$$

If we take $\delta_N = \dfrac{1}{2\nu}$, then

$$\mu_\nu = \frac{1}{\nu}, \qquad \lambda_{\mu+\nu} = \left(1 - \frac{n}{\nu+1}\right) \frac{2 \sin n\,\delta_N}{n} \quad (n = 1, 2, \cdots, \nu).$$

Further, we have

$$\frac{\lambda\mu}{2} + \sum_{n=1}^{\nu} \lambda_{\mu+n} = \frac{1}{\pi(\nu+1)} \int_{-1/2\nu}^{1/2\nu} K_\nu(u)\, du < 1 \text{ and } > \frac{1}{\pi^2}.$$

Since,

$$K_\nu(u) \geqslant \frac{\nu^2}{\pi} \quad \text{for} \quad 0 \leqslant u \leqslant \delta_N.$$

Hence,

$$\frac{1}{\pi} < \Lambda_M - \Lambda_{N-1} < 2.$$

§ 4. Proof of the Theorem: Let S_M denote the Mth partial sum of Fourier series, where

$$S_M(x) - f(x) = \frac{1}{\pi} \int_0^\pi \phi(t) \frac{\sin(M + \frac{1}{2})t}{\sin \frac{1}{2} t} dt$$

on account of the regularity of harmonic method of summation we need only to prove that

$$t_M = \frac{1}{\pi \log M} \sum_{k=0}^{M} \int_0^\pi \phi(t) \frac{\sin(M - k + \frac{1}{2})t}{(k+1) \sin \frac{1}{2} t} dt$$

$$= O(1). \qquad \ldots(4.1)$$

We get

$$t_M = \frac{1}{\pi \log M} \left(\int_0^{\pi/\mu} + \int_{\pi/\mu}^{\pi} \frac{\phi(t)}{\sin \frac{1}{2} t} \sum_{k=0}^{M} \frac{\sin(M - k + \frac{1}{2})t}{k+1} dt \right.$$

$$= \sigma'_M + \sigma''_M, \quad \text{say.}$$

Now using (3.1)

$$\sigma'_M = \frac{1}{\pi \log M} \int_0^{\pi/\mu} |\phi(t)| \, O(M \log M) \, dt$$

$$= O(1).$$

From (3.3) we have

$$\sigma''_M = \frac{1}{\pi \log M} \left[\int_{\pi/\mu}^{\pi} \frac{\phi(t)}{\sin \frac{1}{2} t} \sin(M + 3/2) t \log \frac{1}{2 \sin \frac{1}{2} t} dt \right.$$

$$\left. + \int_{\pi/\mu}^{\pi} \frac{\phi(t)}{\sin \frac{1}{2} t} \left(\frac{\pi}{2} - \frac{t}{2} \right) \cos(M + 3/2) t \, dt - \int_{\pi/\mu}^{\pi} \frac{\phi(t)}{\sin \frac{1}{2} t} R_M(t) \, dt \right]$$

$$= I_1 + I_2 - I_3, \quad \text{say.}$$

We shall first prove I_3. From (3.2) we have

$$I_3 = O\left(\frac{1}{M \log M}\right) \int_{\pi/\mu}^{\pi} \frac{|\phi(t)|}{t^2} dt$$

$$= O\left(\frac{1}{M \log M}\right) \left\{\left[\frac{\Phi(t)}{t^2}\right]_{\pi/\mu}^{\pi} + 2 \int_{\pi/\mu}^{\pi} \frac{\Phi(t)}{t^3}\, dt\right\}$$

$$= O(1). \qquad \ldots(4.2)$$

Using the relation (3·5), we get

$$I_1 = \frac{1}{\pi \log M} \int_{\pi/\mu}^{\pi} \phi(t) \frac{\sin(M + 3/2)t}{\sin \frac{1}{2} t} \log(1/t)\, dt + o(1)$$

$$= \frac{1}{\pi \log M(\Lambda_M - \Lambda_N)} \int_{\pi/\mu}^{\pi} \frac{\phi(t)}{\sin \frac{1}{2} t} \log(1/t) \sum_{n=N}^{M} \lambda_n \sin\left(n + \frac{1}{2}\right) t\, dt$$

$$+ \frac{1}{\pi(\Lambda_M - \Lambda_N) \log M} \int_{\pi/\mu}^{\pi} \phi(t) \log(1/t) \sum_{n=N+1}^{M+1} (\Lambda_{n-1} - \Lambda_{N-1})$$

$$\times \cos nt\, dt + o(1)$$

$$= I_{1\cdot 1} + I_{1\cdot 2} + o(1), \text{ say.}$$

Further, using the second mean value theorem

$$\int_{\pi/\mu}^{\pi} \phi(t) \log 1/t \cos nt\, dt$$

$$= \log \frac{\mu}{\pi} \int_{\pi/\mu}^{\eta} \phi(t) \cos nt\, dt \;\; (\pi/\mu < n < \pi)$$

$$= \log \mu \left[\int_0^{\eta} \phi(t) \cos nt\, dt - \int_0^{\eta/\mu} \phi(t) \cos nt\, dt \right]$$

$$= \log \mu \left[\int_0^{\eta} \phi(t) \cos nt\, dt + \int_0^{\pi/\mu} |\phi(t)|\, dt \right]$$

$$= \log \mu \left[\int_0^{\eta} \phi(t) \cos nt\, dt + o(1/\mu) \right]$$

$$= \log \mu \int_0^{\eta} \phi(t) \cos nt\, dt + o\left(\frac{\log \mu}{\mu}\right).$$

From the definition of $\phi(t)$, it is clear that its Fourier coefficients are of the same order as those of $f(t)$. Therefore, by virtue of (2·5)

$$I_{1\cdot 2} \to 0.$$

Further, let

$$L_\nu(t) = \frac{1}{\nu + 1} \int_{t - 1/2^\nu}^{t + 1/2^\nu} K_\nu(u)\, du$$

then

$$I_{1\cdot 1} = \frac{1}{\pi(\Lambda_M - \Lambda_N)\log M} \int_{\pi/\mu}^{\pi} \frac{\phi(t)}{\sin\frac{1}{2}t} \sin\left(\mu + \frac{1}{2}\right) t \log(1/t) L_\nu(t)\, dt$$

$$= \frac{1}{\pi(\Lambda_M - \Lambda_N)\log M} \int_{\pi/\mu}^{\pi} \phi(t) \frac{\sin \mu t}{t/2} \log(1/t) L_\nu(t)\, dt + o(1)$$

$$= \frac{1}{\pi(\Lambda_M - \Lambda_N)\log M} \left(\int_{\pi/\mu}^{\pi/\nu} + \int_{\pi/\nu}^{\pi} \right) \phi(t) \frac{\sin \mu t}{t/2}$$

$$\times \log(1/t) L_\nu(t)\, dt + o(1)$$

$$= J_1 + J_2 + o(1), \text{ say.}$$

Now

$$|J_2| \leqslant \frac{A}{\log M} \int_{\pi/\nu}^{\pi} \frac{|\phi(t)|}{t} \log 1/t\, dt \int_{t-1/2\mu}^{t+1/2\nu} \frac{du}{\nu u^2}$$

$$\leqslant \frac{A \log \nu}{\log M \nu^2} \int_{\pi/\nu}^{\xi} \frac{|\phi(t)|}{t^3}\, dt \qquad (\pi/\nu < \xi < \pi)$$

$= o(1)$, where A is an arbitrary constant

and

$$J_1 \leqslant \frac{A}{\log M} \int_{\pi/\mu}^{\pi/\nu} \phi(t) \frac{\sin \mu t}{t} \log(1/t) L_\nu(t)\, dt$$

$$= \frac{A}{\log M} \sum_{k=0}^{[\mu/\nu]} (-1)^k \int_{\pi/\mu}^{2\pi/\mu} \frac{\phi(t + (k\pi/\mu))}{t + \frac{k\pi}{\mu}} \log \frac{1}{t + \frac{k\pi}{\mu}} L_\nu\left(t + \frac{k\pi}{\mu}\right)$$

$$\times \sin \mu t\, dt$$

$$= \frac{A}{\log M} \sum_{k=1}^{[\mu/2\nu]} \int_{\pi/\mu}^{2\pi/\mu} \left[\frac{\phi(t + (2k\pi/\mu))}{\left(t + \frac{2k\pi}{\mu}\right)} \log \frac{1}{t + (2k\pi/\mu)} L_\nu(t + (2k\pi/\mu)) \right.$$

$$\left. - \frac{\phi\left(t + \frac{2k-1}{\mu}\pi\right)}{t + \frac{2k-1}{\mu}\pi} \log \frac{1}{\left(t + \frac{2k-1}{\mu}\pi\right)} L_\nu\left(t + \frac{(2k-1)}{\mu}\pi\right) \right]$$

$$\times \sin \mu t\, dt$$

$$= \frac{A}{\log M} \sum_{k=1}^{[\mu/2\nu]} \Bigg[\int_{\pi/\mu}^{2\pi/\mu} \frac{\phi\left(t+\frac{2k\pi}{\mu}\right)-\phi\left(t+\frac{(2k-\pi)}{\mu}\right)}{t+\frac{2k\pi}{\mu}}$$

$$\times \log \frac{1}{t+\frac{2k\pi}{\mu}} L_\nu\left(t+\frac{2k\pi}{\mu}\right)\sin \mu t\, dt$$

$$-\frac{\pi}{\mu}\int_{\pi/\mu}^{2\pi/\mu} \frac{\phi\left(t+\frac{2k-1}{\mu}\pi\right)}{\left(t+\frac{2k\pi}{\mu}\right)\left(t+\frac{2k-1}{\mu}\pi\right)} \log \frac{1}{\left(t+\frac{2k\pi}{\mu}\right)}$$

$$\times L_\nu\left(t+\frac{2k\pi}{\mu}\right)\sin \mu t\, dt$$

$$+\int_{\pi/\mu}^{2\pi/\mu} \frac{\phi\left(t+\frac{2k-1}{\mu}\pi\right)}{\left(t+\frac{2k-1}{\mu}\pi\right)} \log\left(1-\frac{\pi/\mu}{t+\frac{2k\pi}{\mu}}\right)$$

$$\times L_\nu\left(t+\frac{2k\pi}{\mu}\right)\sin \mu t\, dt$$

$$+\int_{\pi/\mu}^{2\pi/\mu} \frac{\phi\left(t+\frac{2k-1}{\mu}\pi\right)}{\left(t+\frac{2k-1}{\mu}\pi\right)} \log \frac{1}{\left(t+\frac{2k-1}{\mu}\pi\right)}$$

$$\times \left\{L_\nu\left(t+\frac{2\pi k}{\mu}\right)-L_\nu\left(t+\frac{2k-1}{\mu}\pi\right)\right\}\sin \mu t\, dt \Bigg]$$

$$= L_1 - L_2 + L_3 + L_4, \quad \text{say.}$$

Since

$$L_\nu\left(t+\frac{2k\pi}{\mu}\right)$$

is bounded, the condition (2.4) implies

$$L_1 = o\,(1).$$

Since

$$K_\nu(u) \leqslant \nu^2, \quad L_\nu\left(t+\frac{2\pi k}{\mu}\right)$$

is bounded for all v and k then

$$L_2 \leqslant \frac{A}{\log M} \sum_{k=1}^{[\mu/2\nu]} \frac{\mu}{k^2} \log \frac{\mu}{k} \int_{\pi/\mu}^{2\pi/\mu} \left| \phi\left(t + \frac{2k-1}{\mu}\pi\right)\right| dt$$

$$= \frac{A\mu}{\log M} \sum_{k=1}^{[\mu/2\nu]} \frac{\log(\mu/k)}{k^3} \int_{\pi/\mu}^{2k\pi/\mu} |\phi(u)|\, du$$

$$+ \frac{A\mu}{\log M} \left(\frac{\nu}{\mu}\right)^2 \log \nu \int_{\pi/\mu}^{\pi/\nu} |\phi(u)|\, du$$

$$= \frac{A\mu}{\log M} \sum_{k=1}^{[\mu/2\nu]} \frac{\log(\mu/k)}{k^3} o(k/\mu) + o(1)$$

$$= o(1)$$

and

$$L_3 \leqslant \frac{A}{\log M} \sum_{k=1}^{[\mu/2\nu]} \frac{\mu}{k} \log\left(1 - \frac{1}{2(k+1)}\right) \int_{\pi/\mu}^{2\pi/\mu} \left| \phi\left(t + \frac{2k-1}{\mu}\pi\right)\right| dt$$

$$\leqslant \frac{A\mu}{\log M} \sum_{k=1}^{[\mu/2\nu]} \frac{1}{k^2} \int_{\pi/\mu}^{2\pi/\mu} \left| \phi\left(t + \frac{2k-1}{\mu}\pi\right)\right| dt$$

$$= o(1).$$

By the fact (Izumi and Izumi 1968) that

$$\left| L_\nu\left(t + \frac{2k\pi}{\mu}\right) - L_\nu\left(t + \frac{2k-1}{\mu}\pi\right)\right| \leqslant \frac{A\nu}{\mu}$$

it can be proved that

$$L_4 = o(1).$$

Using Riemann-Lebesque theorem and (3.6) we have

$$I_2 = \frac{1}{2\pi \log M} \int_{\pi/\mu}^{\pi} \frac{\phi(t)}{\sin \frac{1}{2} t} (\pi/2 - t/2) \cos(M + \tfrac{3}{2})\, t\, dt$$

$$= \frac{1}{4 \log M} \int_{\pi/\mu}^{\pi} \frac{\phi(t)}{\sin t/2} \cos(M + \tfrac{3}{2})\, t\, dt - \frac{1}{4\pi \log M} \int_{\pi/\mu}^{\pi} \frac{t\phi(t)}{\sin \frac{1}{2} t}$$

$$\times \cos\left(M + \frac{3}{2}\right) t\, dt$$

$$= \frac{1}{4 \log M} \int_{\pi/\mu}^{\pi} \frac{\phi(t)}{\sin \frac{1}{2} t} \cos (M + 3/2)\, t + dt + o(1)$$

$$= \frac{1}{2 \log M (\Lambda_M - \Lambda_N)} \int_{\pi/\mu}^{\pi} \frac{\phi(t)}{\sin \frac{1}{2} t} \sum_{n=N}^{M} \lambda_n \cos\left(n + \frac{1}{2}\right) t\, dt$$

$$+ \frac{1}{2 \log M (\Lambda_M - \Lambda_N)} \int_{\pi/\mu}^{\pi} \phi(t) \sum_{n=N+1}^{M+1} (\Lambda_{n-1} - \Lambda_{N-1})$$

$$\times \sin nt\, dt + o(1)$$

$$= I_{2\cdot 1} + I_{2\cdot 2} + o(1), \text{ say.}$$

Proceeding as in the proof of $I_{1\cdot 2}$, it can be proved that $I_{2\cdot 2} \to 0$, and

$$I_{2\cdot 1} = \frac{1}{2 \log M (\Lambda_M - \Lambda_N)} \int_{\pi/\mu}^{\pi} \frac{\phi(t) \cos (\mu + \frac{1}{2})\, t}{\sin \frac{1}{2} t} L_\nu(t)\, dt$$

$$= \frac{1}{2 \log M (\Lambda_M - \Lambda_N)} \int_{\pi/\mu}^{\pi} \phi(t) \frac{\cos \mu t}{\frac{1}{2} t} L_\nu(t)\, dt + o(1)$$

$$= \frac{1}{2 \log M (\Lambda_M - \Lambda_N)} \left(\int_{\pi/\mu}^{\pi/\nu} + \int_{\pi/\nu}^{\pi} \right) \frac{\phi(t) \cos \mu t}{t/2} L_\nu(t)\, dt + o(1)$$

$$= K_1 + K_2 + o(1), \text{ say.}$$

Since $L_\nu(t)$ is bounded,

$$|K_2| \leqslant \frac{A}{\log M} \int_{\pi/\nu}^{\pi} \frac{|\phi(t)|}{t}\, dt \int_{t - \delta_N}^{t + \delta_N} \frac{du}{\nu u^2}$$

$$\leqslant \frac{A}{\nu^2 \log M} \int_{\pi/\nu}^{\pi} \frac{|\phi(t)|}{t^3}\, dt$$

$$= o(1).$$

$$K_1 \leqslant \frac{A}{\log M} \int_{\pi/\mu}^{\pi/\nu} \phi(t) \frac{\cos \mu t}{t} L_\nu(t)\, dt$$

$$= \frac{A}{\log M} \sum_{k=0}^{[\mu/\nu]} (-1)^k \int_{\pi/\mu}^{2\pi/\mu} \frac{\phi(t + k\pi/\mu)}{(t + k\pi/\mu)} \cos \mu t\, L_\nu\left(t + \frac{k\pi}{\mu}\right) dt$$

$$= \frac{A}{\log M} \sum_{k=1}^{[\mu/2\nu]} \int_{\pi/\mu}^{2\pi/\mu} \left[\frac{\phi(t + 2k\pi/\mu)}{(t + 2k\pi/\mu)} L_\nu(t + 2k\pi/\mu) \right.$$

$$\left. - \frac{\phi\left(t + \frac{(2k-1)\pi}{\mu}\right)}{t + \frac{(2k-1)\pi}{\mu}} L_\nu\left(t + \frac{(2k-1)\pi}{\mu}\right) \right] \cos \mu t \, dt$$

$$= \frac{A}{\log M} \sum_{k=1}^{[\mu/2\nu]} \left[\int_{\pi/\mu}^{2\pi/\mu} \frac{\phi(t + 2k\pi/\mu) - \phi\left(t + \frac{2k-1}{\mu}\pi\right)}{t + \frac{2k\pi}{\mu}} \right.$$

$$\times L_\nu\left(t + \frac{2k\pi}{\mu}\right) \cos \mu t \, dt$$

$$+ \frac{\pi}{\mu} \int_{\pi/\mu}^{2\pi/\mu} \frac{\phi\left(t + \frac{2k-1}{\mu}\pi\right)}{\left(t + \frac{2k\pi}{\mu}\right)\left(t + \frac{(2k-1)\pi}{\mu}\right)}$$

$$\times L_\nu\left(t + \frac{2k\pi}{\mu}\right) \cos \mu t \, dt$$

$$+ \int_{\pi/\mu}^{2\pi/\mu} \frac{\phi\left(t + \frac{2k-1}{\mu}\pi\right)}{\left(t + \frac{2k-1}{\mu}\pi\right)} \left\{ L_\nu\left(t + \frac{2k\pi}{\mu}\right) \right.$$

$$\left. \left. - L_\nu\left(t + \frac{(2k-1)\pi}{\mu}\right) \right\} \cos \mu t \, dt \right]$$

$$= M_1 + M_2 + M_3, \text{ say.}$$

$$L_\nu\left(t + \frac{2k\pi}{\pi}\right)$$

is bounded and using the second mean value theorem, we have

$$|M_1| = \frac{A}{\log \mu \log M} \left| \sum_{k=1}^{[\mu/2\nu]} \int_{\pi/\mu'}^{2\pi/\mu} \frac{\phi\left(t + \frac{2k\pi}{\mu}\right) - \phi\left(t + \frac{2k-1}{\mu}\pi\right)}{t + \frac{2k\pi}{\mu}} \right.$$

$$\left. \times \log 1/t \, dt \right| \quad (\pi/\mu < \pi/\mu' < 2\pi/\mu)$$

$$= o(1).$$

Proceeding as in the proof of L_2 and L_4, it can be seen that

$$M_2 = o(1) \quad \text{and} \quad M_3 = o(1).$$

This completes the proof of our theorem.

References

Hille, E., and Tamarkin, J. D. (1932). On the summability of Fourier series. *Trans. Am. math. Soc.*, **34**, 757–83.

Izumi, M., and Izumi, S. (1968). Some convergence criteria for Fourier series. *J. Indian math. Soc.*, **32**, 355–79.