ON SUBCLASSES OF SCHLICHT MAPPING

by R. N. DAS *and* P. SINGH, *Department of Mathematics, Indian Institute of Technology, Kanpur*

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A number of subclasses of Schlicht functions, such as the classes of star-like functions, convex functions and close to convex functions w.r.t. symmetric points in $|z| < 1$ have been studied. The behaviours of certain integral operators, applied on the members of these classes, are found to be within these classes, thus giving a closer view of these classes regarding their geometric aspects and other properties.

1. INTRODUCTION

Let S denote the class of functions which are analytic and Schlicht in the unit disc $E(|z| < 1)$, satisfying the normalization $f(0) = 0 = f'(0) - 1$. Let S^*, C, k be the subfamilies of S, whose members map E on to domains which are star-like w.r.t. origin, convex and close to convex respectively, then as was observed by Kaplan (1952), we have $C \subseteq S^* \subseteq K \subseteq S$. Let P denote the class of functions $p(z)$, which are analytic, $p(0) = 1$ and satisfy $\operatorname{Re} p(z) > 0$ for $|z| < 1$. The following results are well known (Robertson 1936).

If $f(z) \in S$, then

(i) $f(z) \in S^*$ if and only if $\dfrac{zf'(z)}{f(z)} \in P$

(ii) $f(z) \in C$ if and only if $1 + \dfrac{zf''(z)}{f'(z)} \in P$

(iii) $f(z) \in C$ if and only if $zf'(z) \in S^*$

Our aim is to look after the class of functions which need not be convex, but which form a subclass of the class K close to the convex functions introduced by Kaplan (1952). It is well known that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is Schlicht and convex in $|z| < 1$, then $|a_n| \leq 1$. For the class of functions $f(z)$ which are close to convex in $|z| < 1$, the inequalities $|a_n| \leq n$ hold. We now consider another class of functions, which are also close to convex in $|z| < 1$, but bot necessarily convex, for which the estimate $|a_n| \leq 1$ is true. This observation leads us to define certain subclasses of S—star-like, convex and close to convex w.r.t. symmetric points. Sakaguchi (1959) defined the Schlicht functions which are star-like w.r.t. symmetric points, having a nice geometric feature, as follows:

Definition (Sakaguchi)—Let $f(z)$ be analytic in $|z| < 1$ and suppose that for every $r \to 1$ $(r < 1)$ and every ξ on $|z| = r$, the angular velocity of $f(z)$ about the point $f(-\xi)$ is positive at $z = \xi$, as z traverses the circle $|z| = r$ in the positive direction

i.e.,

$$\text{Re}\, \frac{zf'(z)}{f(z) - f(-\xi)} > 0 \quad \text{for} \quad z = \xi, \quad |\xi| = r. \qquad \text{...(1.1)}$$

In this case, $f(z)$ is called star-like w.r.t. symmetric points in $|z| < 1$ and the class of functions, satisfying (1.1) can be denoted by S_s^*. We note the following results of Sakaguchi (1959) for the members of the class S_s^*.

Theorem A—Let $f(z)$ be analytic in $|z| < 1$, having the representation $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then a necessary and sufficient condition for $f(z)$ to belong to S_s^* [i.e., $f(z)$ is Schlicht and star-like w.r.t. symmetric points in $|z| < 1$] is that

$$\text{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, \quad |z| < 1. \qquad \text{...(1.2)}$$

Theorem B—Let $f(z)$ be Schlicht and star-like w.r.t. symmetric points in $|z| < 1$, having the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then

$$|a_n| \leq 1 \text{ for all } n \geq 2 \qquad \text{...(1.3)}$$

and, the equality is attained for the function $f(z) = z/(1 + \varepsilon z)$, $|\varepsilon| = 1$.

Remark 1: The class of the functions Schlicht and star-like w.r.t. symmetric point obviously includes the classes of convex functions and odd star-like functions w.r.t. the origin.

The result stated below is due to Robertson (1961) and relates to the class S_s^* regarding its geometry and other aspects.

Theorem C—Let the function $(1 - t) f(z) + t f(-z)$ be subordinate to the Schlicht, analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $|z| < 1$ for an interval $0 \leq t \leq t_0$, then the relation (1.2) holds and the vector $f(z) - f(-z)$ turns continuously in one direction as z traverses each circle $|z| = r < 1$. $f(z)$ is close to convex in $|z| < 1$.

This indicates the manner in which the members of the class S_s^* are generated with the help of the subordination principle and we are in a position to write $C \subset S_s^* \subset K \subset S$. Our purpose is to extend the results of Sakaguchi to other classes of functions w.r.t. symmetric points in $|z| < 1$ and other materials. We start with the following definition:

Definition 1—Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $|z| < 1$ and suppose that for every $r \to 1$ $(r < 1)$ and every ξ on $|z| = r$, the following relation holds :

$$\text{Re}\left(\frac{z\,(zf'(z))'}{zf'(z) + \xi f'(-\xi)}\right) > 0 \quad \text{for} \quad z = \xi, |\xi| = r. \qquad ...(1.4)$$

Here, $f(z)$ is referred to as the convex function w.r.t. symmetric points in $|z| < 1$ and the class of functions satisfying (1.4) can be denoted by C_s.

2. Properties of the Class C_s

Theorem 1—Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $|z| < 1$, then a necessary and sufficient condition for $f(z)$ to be Schlicht and convex w.r.t. symmetric points in $|z| < 1$ is that

$$\text{Re}\left(\frac{(zf'(z))'}{(f(z) - f(-z))'}\right) > 0, \quad |z| < 1. \qquad ...(2.1)$$

Proof: We first suppose that (2.1) holds, then $f(z)$ is evidently convex w.r.t. symmetric points and, therefore, it is sufficient to show that $f(z)$ is Schlicht in $|z| < 1$. Using (2.1) and the result obtained on substituting $-z$ for z in (2.1), it follows that the function $f(z) - f(-z)$ becomes Schlicht and convex in $|z| < 1$, so that the functions $z\,(f(z) - f(-z))'$ becomes Schlicht and star-like in $|z| < 1$. Therefore, in view of (2.1), it follows that $zf'(z)$ is close to convex for $|z| < 1$ and hence $zf'(z)$ is Schlicht there. Now, consider the function $g(z) = zf'(z)$, then

$$f(z) = \int_0^z \frac{g(t)}{t}\, dt$$

and, since $g(z) \in K$, we have finally $f(z) \in S$, as required. The necessary part follows easily from the relevant definition with simple arguments.

Remark 2: From the above theorem and the earlier discussion, it follows that

$$f(z) \in C_s \text{ if and only if } zf'(z) \in S_s^* \qquad ...(2.2)$$

The coefficient estimates for the functions of the class C_s is given by

Theorem 2—Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be Schlicht and convex w.r.t. symmetric points in $|z| < 1$, then

$$|a_n| \leq 1/n \text{ for all } n \geq 2 \qquad ...(2.3)$$

and the equality holds for the functions

$$f(z) = (1/\varepsilon) \log(1 + \varepsilon z), \quad |\varepsilon| = 1. \qquad ...(2.4)$$

PROOF: The result follows on applying the procedure of Theorem B and the relation (2.2).

Next, we define the notion of close to convex functions w.r.t. symmetric points in $|z| < 1$ as follows:

Definition 2—Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic for $|z| < 1$, then $f(z)$ is said to be close to convex function w.r.t. symmetric points in $|z| < 1$, if there exists a function $g(z)$, Schlicht and convex w.r.t. symmetric points for $|z| < 1$, such that

$$\mathrm{Re}\left(\frac{f'(z)}{g'(z) + g'(-z)}\right) > 0, \ |z| < 1.$$

The class of such functions can be denoted by K_s. Using (2.2), it means that $f(z) \in K_s$, if there exists a $h(z) \in S_s^*$, such that

$$\mathrm{Re}\left(\frac{zf'(z)}{h(z) - h(-z)}\right) > 0, \quad |z| < 1.$$

3. PROPERTIES OF THE CLASS K_s

Proposition 1—Our aim is now to show that "every close to convex function w.r.t. symmetric points in $|z| < 1$ is Schlicht there". Let $f(z) \in K_s$ w.r.t. $g(z) \in C_s$. Since $g(z) \in C_s$, $G(z) = (1/2)[g(z) - g(-z)] \in C$ and that $zG'(z) \in S^*$, which shows that $f(z) \in K \subseteq S$.

The class K_s obviously includes several familiar subclasses of Schlicht functions, i.e., convex functions w.r.t. symmetric points, the star-like functions w.r.t. symmetric points, the functions whose derivatives have positive real parts in $|z| < 1$, etc. To see the coefficient bounds for the members of the class K_s, we have the following:

Theorem 3—If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K_s$ w.r.t. some $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S_s^*$, then the estimates (1.3) hold and the equality is attained for the function $z/(1 + \varepsilon z)$, $|\varepsilon| = 1$.

PROOF: According to the hypothesis,

$$\mathrm{Re}\left(\frac{zf'(z)}{g(z) - g(-z)}\right) > 0, \ |z| < 1.$$

Let

$$F(z) = \frac{2zf'(z)}{g(z) - g(-z)} = 1 + \sum_{n=1}^{\infty} c_n z^n \qquad \text{...(3.1)}$$

so that Re $F(z) > 0$, $|z| < 1$ and for which the estimate $|c_n| \leq 2, n \geq 1$ is known. On using the series expansion in (3.1),

$$z + \sum_{n=2}^{\infty} na_n z^n = \left(\sum_{n=1}^{\infty} b_{2n-1} z^{2n-1}\right)\left(1 + \sum_{n=1}^{\infty} c_n z^n\right) \qquad \ldots(3.2)$$

and then equating the corresponding coefficients from both sides in (3.2) and using the estimates $|b_n| \leq 1$, $n \geq 2$, the result follows.

If $f(z) \in S_s^*$, C_s, K_s respectively, then we are motivated onwards to see whether the functions

$$(2/z)\int_0^z f(t)\,dt \quad \text{and} \quad (1/2)\int_0^z \frac{f(t) - f(-t)}{t}\,dt$$

also belong to these classes respectively or not ! We need the following lemmas:

Lemma 1—If $f(z) \in S_s^*$, then

$$F(z) = (1/2)[f(z) - f(-z)] \in S^*. \qquad \ldots(3.3)$$

Lemma 2—If $(z) \in C_s$, then

$$F(z) = (1/2)[f(z) - f(-z)] \in C. \qquad \ldots(3.4)$$

Lemma 3 (*Libera* 1965)—If $f(z) \in S^*$, then $F(z) = \int_0^z f(t)\,dt$ is 2 valently star-like function w.r.t. the origin for $|z| < 1$.

Lemma 4 (*Libera* 1965)—If N and D be analytic in $|z| < 1$ and $N(0) = D(0) = 0$, D maps $|z| < 1$ onto a many-sheeted region, which is star-like w.r.t the origin and $N'/D' \in P$, then necessarily we have $N/D \in P$. The first two lemmas are trivial and Lemma 1 is an improvement of a result of Singh (1970), which proves it for a smaller class $C \subseteq S_s^*$.

Lemma 5—If $f(z) \in K_s$ w.r.t. $g(z) \in C_s$, then

$$\text{Re}\left(\frac{f(z) - f(-z)}{g(z) - g(-z)}\right) > 0, \quad |z| < 1.$$

Lemma 6—If $zf'(z)$ is Schlicht and close to convex w.r.t. symmetric points in $|z| < 1$, then $f(z)$ is also Schlicht and close to convex w.r.t. symmetric points there.

The proofs of Lemmas 5 and 6 follow as an application of Lemma 4 or otherwise following the procedure of a theorem due to Singh (1970).

We observe that the last two lemmas are analogous to the results of Singh (1970) and Sakaguchi (1969) respectively.

4. Main Theorems Related to the Class S_s^*, C_s and K_s

Theorem 4—If $f(z) \in S_s^*$ and $F(z)$, as defined by $F(z) = (2/z) \int_0^z f(t)\,dt$, then $F(z) \in S_s^*$ as well.

Proof: We observe that

$$\frac{zf'(z)}{f(z) - f(-z)} = \frac{(z^2 F'(z))'}{[z\{F(z) - F(-z)\}]}. \qquad \text{...(4.1)}$$

Since $f(z) \in S_s^*$, the real part of the R.H.S. of (4.1) is positive for $|z| < 1$. But, $z[F(z) - F(-z)]$ maps $|z| < 1$ onto a many-sheeted region, star-like w.r.t. the origin, as a consequence of Lemmas 1 and 3. Finally, on applying Lemma 4, the result follows.

Corollary—If $f(z) \in S_s^*$, then $\int_0^z (2/t^2)\,[\int_0^t f(p)\,dp]\,dt$ defines a member of the class C_s.

Theorem 5—If $f(z) \in C_s$ and $F(z) = (2/z) \int_0^z f(t)\,dt$, then $F(z) \in C_s$.

Proof: Let $g(z) = zf'(z)$, so that $g(z) \in S_s^*$ and using Theorem 4, we have $G(z) = (2/z) \int_0^z g(t)\,dt \in S_s^*$. But $G(z) = zF'(z)$ and hence $F(z) \in C_s$.

Theorem 6—If $f(z) \in K_s$ w.r.t. $g(z) \in S_s^*$, $F(z) = (2/z) \int_0^z f(t)\,dt$, $G(z) = (2/z) \int_0^z g(t)\,dt$, then $F(z) \in K_s$ w.r.t. $G(z) \in S_s^*$ as well.

Proof: Since $g(z) \in S_s^*$, we have $G(z) \in S_s^*$. Also,

$$\frac{zf'(z)}{g(z) - g(-z)} = \frac{[zF'(z)]}{[G(z) - G(-z)]} \qquad (4.2)$$

Using the hypothesis, the real part of the R.H.S. of (4.2) is positive for $|z| < 1$. But, $G(z) - G(-z)$ is 2-valently star-like w.r.t. origin, as a consequence of Lemmas 1 and 3. Finally, on using Lemma 4, the result follows.

The last three theorems are analogous to the results of Libera (1965). The problems converse to these theorems remain open.

Theorem 7—If $f(z) \in S_s^*$, then $F(z)$ is defined by

$$F(z) = (1/2) \int_0^z \frac{f(t) - f(-t)}{t}\,dt \in S_s^* \qquad \text{...(4.3)}$$

Proof: Since $f(z) \in S_s^*$ on using Lemma 1 and the fact that $f(z) = z\phi'(z)$ with $\phi(z) \in C_s$, we have

$$F'(z) = (1/2)[\phi'(z) + \phi'(-z)], \text{ so that } F(z) = (1/2)[\phi(z) - \phi(-z)].$$

Also, since $C \subseteq S_s^*$, on using Lemma 2 to $\phi(z)$, we conclude that $F(z) \in S_s^*$.

Theorem 8—If $g(z) \in C_s$, then the function $G(z)$ defined by

$$G(z) = (1/2) \int_0^z \frac{g(t) - g(-t)}{t} \, dt \in C_s. \quad \ldots(4.4)$$

PROOF: Since $g(z) \in C_s$, we have $(1/2)[g(z) - g(-z)] \in C \subseteq S^*$. Also,

$$\operatorname{Re}\left(\frac{[zG'(z)]'}{G'(z) + G'(-z)}\right) = (1/2) \operatorname{Re}\left(\frac{z[g'(z) + g'(-z)]'}{g(z) - g(-z)}\right) \quad \ldots(4.5)$$

which shows that $G(z) \in C_s$.

Theorem 9—If $f(z) \in K_s$ w.r.t. $g(z) \in C_s$, $F(z)$ and $G(z)$ are as given in (4.3) and (4.4), then $F(z) \in K_s$ w.r.t. $G(z) \in C_s$.

PROOF: From the representation of $F(z)$ and $G(z)$, we have

$$(1/2)\left(\frac{f(z) - f(-z)}{g(z) - g(-z)}\right) = \frac{F'(z)}{G'(z) + G'(-z)} \quad \ldots(4.6)$$

Consequently, the result follows on using the hypothesis and Lemma 5.

The last three theorems are analogous to the results of Singh (1970).

5. A Generalized View for the Class S_s^*

To have a generalization of the condition for the functions to be star-like w.r.t. symmetric points in $|z| < 1$, we have the following:

Theorem 10—Suppose $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $|z| < 1$ and that for a positive integer k, the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{\sum_{n=0}^{k-1} \frac{f(\varepsilon^n z) - f(-\varepsilon^n z)}{\varepsilon^n}}\right) > 0, \quad |z| < 1 \text{ holds, where } \varepsilon = e^{2\pi i/k} \quad \ldots(5.1)$$

then $f(z)$ is Schlicht and close to convex function w.r.t. symmetric points in $|z| < 1$.

PROOF: Substituting $\varepsilon^m z$ for z in (5.1), we have

$$\operatorname{Re}\left(\frac{zf'(\varepsilon^m z)}{\sum_{n=0}^{k-1} \frac{f(\varepsilon^n z) - f(-\varepsilon^n z)}{\varepsilon^n}}\right) > 0, |z| < 1 \quad \ldots(5.2)$$

where m is an integer.

Hence,

$$\operatorname{Re}\left(\frac{\sum_{m=0}^{k-1} zf'(\varepsilon^m z)}{\sum_{n=0}^{k-1} \frac{f(\varepsilon^n z) - f(-\varepsilon^n z)}{\varepsilon^n}}\right) > 0, \quad |z| < 1 \qquad \ldots(5.3)$$

which shows that the function

$$\sum_{n=0}^{k-1} \frac{f(\varepsilon^n z)}{\varepsilon^n} = kz + \cdots$$

is Schlicht and star-like w.r.t. symmetric points for $|z| < 1$. Hence, from (5.1), $f(z)$ is Schlicht and close to convex w.r.t. symmetric points there.

Next, we shall generalize the condition (5.1) and conclude the following:

Theorem 11—If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $|z| < 1$ and put $F_0(z) = f(z)$, $F_1(z) = zf'(z)$, $F_2(z) = z[zf'(z)]'$, $\cdots$ If for a non-negative integer m, we have

$$\operatorname{Re}\left(\frac{zF'_m(z)}{\sum_{n=0}^{k-1} \frac{F_m(\varepsilon^n z) - F_m(-\varepsilon^n z)}{\varepsilon^n}}\right) > 0, \quad |z| < 1 \qquad \ldots(5.4)$$

where k is a positive integer and $\varepsilon = e^{-2\pi i/k}$, then $f(z)$ is Schlicht and close to convex w.r.t. symmetric points in $|z| < 1$.

PROOF: When (5.4) holds, by Theorem 10, it follows that $F_n(z)$ is Schlicht and close to convex w.r.t. symmetric points in $|z| < 1$. Consequently, from Lemma 6, the functions $F_{n-1}(z)$, $F_{n-2}(z)$, $\cdots$ $F_0(z) = f(z)$ are all Schlicht and close to convex w.r.t. symmetric points there.

6. Concept of Order of the Functions in S_s^*, C_s and K_s

The interesting properties of the functions of the classes considered lead us to define the order of such functions. By $S_s^*(\alpha)$, we denote the class of functions $f(z) \in S_s^*$, having the additional property

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > \alpha, \quad |z| < 1, \; 0 \leq \alpha < 1/2. \qquad (6.1)$$

Here, of course, α is referred to as the order of star-like function $f(z)$ w.r.t. symmetric points in $|z| < 1$ and identify $S_s^*(0) = S_s^*$. Similarly, $C_s(\alpha)$ denotes the class of functions $f(z) \in C_s$ together with the property that

$$\operatorname{Re}\left(\frac{(zf'(z))'}{f'(z) + f'(-z)}\right) > \alpha, \quad |z| < 1, \quad 0 \leq \alpha < 1/2 \qquad \ldots(6.2)$$

and identify $C_s(0) = C_s$. Finally, $K_s(\alpha)$ denotes the class of functions $f(z) \in K_s$ w.r.t. some $g(z) \in C_s$ and also satisfying the condition

$$\text{Re}\left(\frac{f'(z)}{g'(z) + g'(-z)}\right) > \alpha, \quad |z| < 1, 0 \leq \alpha < 1/2 \qquad \ldots(6.3)$$

and identify $K_s(0) = K_s$. It follows that $C_s(\alpha) \subseteq S_s^*(\alpha) \subseteq K_s(\alpha)$ for $0 \leq \alpha < 1/2$.

The sharp distortion theorems and radius of convexity for the class $S_s^*(\alpha)$ were earlier obtained by Das and Singh (1977).

References

Das, R. N., and Singh, P. (1977). Radius of convexity for a certain sub-class of close to convex functions. *J. Indian math. Soc.*, **41** (to appear).

Kaplan, W. (1952). Close to convex Schlicht functions. *Mich. math. J.*, **1**, 169–85.

Libera, R. J. (1965). Some classes of regular univalent functions. *Proc. Am. math. Soc.*, **16**, 755–58.

Robertson, M. S. (1936). On the theory of univalent functions. *Ann. Math.*, **37** (2), 374–408.

——— (1961). Applications of the subordination principle of univalent functions. *Pacif. J. Math.*, **11**, 315–24.

Sakaguchi, K. (1959). On a certain univalent mapping. *J. math. Soc. Japan*, **11**, 72–80.

Singh, R. (1970). Some classes of regular univalent function. *Revista mat. hisp.-am. 4a Ser.*, Tomo, 29, **30**, 109–14.