AN ELECTROSTATIC PROBLEM OF TWO STAGGERED PARALLEL STRIPS IN A BOUNDED SPACE

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We present here the solution of the electrostatic problem of two parallel non-coplanar conducting infinite strips charged to prescribed constant potentials in the space bounded by two grounded planes which are parallel to the strips. The pair of simultaneous governing Fredholm integral equations of the first kind are solved by the regular perturbations technique, when the distance between the parallel strips is large and the bounding planes are far away from the strips. Approximate expressions for the total charge densities per unit length of the strips as well as the total charges per unit length of the strips are derived. Finally, by taking appropriate limits, we obtain the corresponding expressions, when the strips are lying in an unbounded space and also derive the expressions for the capacity per unit length of the condenser formed by the two strips. These limiting results agree with the known results.

1. INTRODUCTION

Tranter (1960) solved the two-dimensional electrostatic problem of two equal parallel coplanar conducting infinite strips charged to potentials ± 1 in a free space by a dual series method. Srivastava and Gupta (1971) further studied the perturbation in the charge density of the two strips, when these are placed symmetrically inside or outside a grounded cylinder by using triple integral equations and finite Hilbert transform techniques (Srivastava and Lowengrub 1970). Goel and Jain (1976) showed how the solutions of the two electrostatic problems discussed by Srivastava and Gupta (1971) can be simplified by using Green's function approach and solved the resulting governing Fredholm integral equation of the first kind by the regular perturbation technique, as explained by Jain and Kanwal (1972, 1975 a, b).

We present here solution of the two-dimensional electrostatic problem of two equal parallel non-coplanar conducting infinite strips charged to prescribed constant potentials in the space bounded by two grounded infinite planes which are parallel to the two strips. The problem is formulated by following the usual Green's function approach and the solution of the problem is reduced to that of two simultaneous Fredholm integral equations of the first kind. These integral equations are further solved by the regular perturbation technique, when the distance between the two strips is large and the bounding planes are far away from the strips.

Finally, we discuss the limiting case when the two strips are lying in an unbounded space and obtain an approximate expression for the capacity per unit length of the condenser formed by two parallel non-coplanar strips.

2. Mathematical Formulation

Let the two equal parallel non-coplanar conducting infinite strips

$$-a_1 \leqslant x \leqslant b_1, \quad y = -d_1, \ -\infty < z < \infty,$$

$$-b_1 \leqslant x \leqslant a_1, \quad y = d_1, \ -\infty < z < \infty$$

be charged to constant potentials V_1 and V_2 in the space bounded by grounded infinite planes $y = \pm f_1$, where $-a_1 < b_1 \leqslant a_1, a_1 > 0$ and $f_1 > d_1 > 0$. We non-dimensionalize all lengths by a_1, where $2a_1$ is the distance between the outer edges of the two strips. Thus, we have to solve the following two-dimensional boundary value problem for the electrostatic potential $V(x, y)$:

$$\nabla^2 V(x, y) = 0, \quad \text{in} \quad D \qquad \ldots(2.1)$$

$$V(x, -d) = V_1, \quad -1 \leqslant x \leqslant b \qquad \ldots(2.2)$$

$$V(x, d) = V_2, \quad -b \leqslant x \leqslant 1 \qquad \ldots(2.3)$$

$$V(x, -f) = 0, \quad -\infty < x < \infty \qquad \ldots(2.4)$$

$$V(x, f) = 0, \quad -\infty < x < \infty \qquad \ldots(2.5)$$

where $b = b_1/a_1$, $d = d_1/a_1$, $f = f_1/a_1$, and D is the entire region of the infinite strips $y = \pm f$, $-\infty < x < \infty$, except the two-line segments

$$-1 \leqslant x \leqslant b, \quad y = -d \quad \text{and} \quad -b \leqslant x \leqslant 1, \quad y = d.$$

3. Solution of the Problem

The integral formula for $V(x, y)$ follows from the usual Green's function approach. Indeed, the function $V(x, y)$ satisfying (2.1), (2.4) and (2.5) is

$$V(x, y) = \int_{-1}^{b} \sigma(x_0, -d)\, G(x, y \mid x_0, -d)\, dx_0 + \int_{-b}^{1} \sigma(x_0, d)\, G(x, y \mid x_0, d)\, dx_0$$

$$= \int_{-1}^{b} \{\sigma(x_0, -d)\, G(x, y \mid x_0, -d) + \sigma(-x_0, d)\, G(x, y \mid -x_0, d)\}\, dx_0 \qquad \ldots(3.1)$$

where $\sigma(x_0, -d)$, $-1 \leqslant x_0 \leqslant b$; $\sigma(x_0, d)$, $-b \leqslant x_0 \leqslant 1$, are the unknown total charge densities per unit length of the two strips and the Green's function G

constructed by applying the conformal mapping theorem as given by Mackie (1965) is

$$G(x, y \mid x_0, y_0) = \frac{-1}{4\pi} \log \left\{ \frac{\cosh[\pi(x - x_0)/2f] - \cos[\pi(y - y_0)/2f]}{\cosh[\pi(x - x_0)/2f] + \cos[\pi(y + y_0)/2f]} \right\} \qquad \text{...(3.2)}$$

The boundary conditions (2.2), (2.3) and the formula (3.1) lead to the following two simultaneous Fredholm integral equations of the first kind for the unknown total charge densities per unit length, $\sigma(x_0, -d)$ and $\sigma(-x_0, d)$, $-1 \leqslant x_0 \leqslant b$.

$$\int_{-1}^{b} \{\sigma(x_0, -d)\, G(x, -d \mid x_0, -d) + \sigma(-x_0, d)\, G(x, -d \mid -x_0, d)\}\, dx_0 = V_1, \; -1 \leqslant x \leqslant b \qquad \text{...(3.3)}$$

$$\int_{-1}^{b} \{\sigma(x_0, -d)\, G(-x, d \mid x_0, -d) + \sigma(-x_0, d)\, G(-x, d \mid -x_0, d)\}\, dx_0 = V_2, \; -1 \leqslant x \leqslant b. \qquad \text{...(3.4)}$$

We solve the two simultaneous integral equations (3.3) and (3.4) when $f \gg d \gg 1$ and $(d/f) = 0\ (1/d)$. Thus, $(1/f) = 0\ (1/d^2)$ and we let $d^2/f = \beta$, where $\beta = 0\,(1)$. Using formula (3.2), we express the values of Green's functions occurring in eqns. (3.3) and (3.4) in powers of the small dimensionless perturbation parameter $(1/d)$ and obtain

$$G(\pm x, \mp d \mid \pm x_0, \mp d) = \frac{-1}{2\pi} \left\{ \log|x - x_0| + \log\frac{\pi}{4f} + \frac{\pi^2 \beta^2}{8d^2} + O\left(\frac{1}{d^4}\right) \right\} \qquad \text{...(3.5)}$$

$$G(\pm x, \mp d \mid \mp x_0, \pm d) = \frac{-1}{2\pi} \left\{ \log\left(\frac{\pi d}{2f}\right) - \frac{\pi^2 \beta^2}{24 d^2} + \frac{(x + x_0)^2}{8d^2} + O\left(\frac{1}{d^4}\right) \right\} \qquad \text{...(3.6)}$$

Adding and subtracting (3.3) and (3.4), we obtain

$$\int_{-1}^{b} \{\sigma(x_0, -d) + \sigma(-x_0, d)\}\, K_1(x, x_0)\, dx_0 = V_1 + V_2, \; -1 \leqslant x \leqslant b \qquad \text{...(3.7)}$$

$$\int_{-1}^{b} \{\sigma(x_0, -d) - \sigma(-x_0, d)\}\, K_2(x, x_0)\, dx_0 = V_1 - V_2, \; -1 \leqslant x \leqslant b \qquad \text{...(3.8)}$$

where

$$K_1(x, x_0) = \frac{-1}{2\pi} \left\{ \log|x - x_0| + \log\left(\frac{\pi^2 d}{8f^2}\right) + \left(\frac{2\pi^2 \beta^2 + 3(x + x_0)^2}{24 d^2}\right) + O\left(\frac{1}{d^4}\right) \right\}, \quad -1 \leqslant x \leqslant b \qquad \text{...(3.9)}$$

$$K_2(x, x_0) = \frac{-1}{2\pi}\left\{\log|x - x_0| - \log 2d + \left(\frac{4\pi^2\beta^2 - 3(x + x_0)^2}{24d^2}\right) + 0\left(\frac{1}{d^4}\right)\right\}, \quad -1 \leqslant x \leqslant b. \quad ...(3.10)$$

The above expansions suggest that the unknown charge densities $[\sigma(x_0, -d) \pm \sigma(-x_0, d)]$ can also be expressed as

$$\sigma_1(x_0) = \sigma(x_0, -d) + \sigma(-x_0, d)$$

$$= \sigma_{10}(x_0) + \frac{1}{d^2}\sigma_{11}(x_0) + O\left(\frac{1}{d^4}\right) \quad ...(3.11)$$

$$\sigma_2(x_0) = \sigma(x_0, -d) - \sigma(-x_0, d)$$

$$= \sigma_{20}(x_0) + \frac{1}{d^2}\sigma_{21}(x_0) + O\left(\frac{1}{d^4}\right) \quad ...(3.12)$$

When we substitute the expansions (3.9) to (3.12) in eqns. (3.7) and (3.8) and equate the coefficients of equal powers of $(1/d)$ on either sides, we obtain the following pairs of simultaneous Fredholm integral equations of the first kind:

$$\int_{-1}^{b} \sigma_{10}(x_0) M_1(x_0, x)\, dx_0 = -2\pi(V_1 + V_2), \; -1 \leqslant x \leqslant b \quad ...(3.13)$$

$$\int_{-1}^{b} \sigma_{11}(x_0) M_1(x_0, x)\, dx_0 = -\int_{-1}^{b} \sigma_{10}(x_0)\left[\frac{\pi^2\beta^2}{12} + \frac{(x + x_0)^2}{8}\right] dx_0,$$

$$-1 \leqslant x \leqslant b, \quad ...(3.14)$$

$$\int_{-1}^{b} \sigma_{20}(x_0) M_2(x_0, x)\, dx_0 = -2\pi(V_1 - V_2), \quad -1 \leqslant x \leqslant b \quad ...(3.15)$$

$$\int_{-1}^{b} \sigma_{21}(x_0) M_2(x_0, x)\, dx_0 = -\int_{-1}^{b} \sigma_{20}(x_0)\left[\frac{\pi^2\beta^2}{6} - \frac{(x + x_0)^2}{8}\right] dx_0,$$

$$-1 \leqslant x \leqslant b \quad ...(3.16)$$

and so on and where

$$M_1(x_0, x) = \log|x - x_0| + \log\left(\frac{\pi^2 d}{8f^2}\right);$$

$$M_2(x_0, x) = \log|x - x_0| - \log(2d). \quad ...(3.17)$$

Eqns. (3.13) - (3.16) can be solved separately to obtain successively the values of $\sigma_{1j}(x_0)$ and $\sigma_{2j}(x_0)$, $j = 1, 2$ by using the substitutions:

$$x_0 = \tfrac{1}{2}b(1 + \cos\theta_0) - \tfrac{1}{2}(1 - \cos\theta_0);$$

$$x = \tfrac{1}{2}b(1 + \cos\theta) - \tfrac{1}{2}(1 - \cos\theta), \; O \leqslant \theta_0, \; \theta \leqslant \pi. \quad ...(3.18)$$

From (3.13), (3.15) and (3.18), it follows that

$$\left(\frac{1+b}{2}\right)\int_0^\pi \sigma_{10}(x_0)\sin\theta_0 N_1(\theta,\theta_0)\,d\theta_0 = -2\pi(V_1+V_2),\quad 0\leqslant\theta\leqslant\pi \qquad ...(3.19)$$

$$\left(\frac{1+b}{2}\right)\int_0^\pi \sigma_{20}(x_0)\sin\theta_0 N_2(\theta,\theta_0)\,d\theta_0 = -2\pi(V_1-V_2),\ 0\leqslant\theta\leqslant\pi \qquad ...(3\cdot20)$$

where

$$N_1(\theta,\theta_0) = M_1(x,x_0) = -2\sum_{n=1}^{\infty}\frac{\cos n\theta\cos n\theta_0}{n} + \lambda_1$$

$$N_2(\theta,\theta_0) = M_2(x,x_0) = -2\sum_{n=1}^{\infty}\frac{\cos n\theta\cos n\theta_0}{n} - \lambda_2$$

$$\lambda_1 = \log\left[\frac{\pi^2(1+b)\,d}{32f^2}\right],\qquad \lambda_2 = \log\left[\frac{8d}{1+b}\right] \qquad ...(3\cdot21)$$

Eqns. (3·19) and (3·20) readily yield

$$\sin\theta_0\,\sigma_{j0}(x_0) = A_j,\qquad j=1,2 \qquad ...(3\cdot22)$$

where the constants A_j are defined by

$$A_1 = -4(V_1+V_2)/(1+b)\lambda_1 \qquad A_2 = 4(V_1-V_2)/(1+b)\lambda_2.$$

Similarly, it follows from eqns. (3·14), (3·16), (3·18) and (3·22)

$$\int_0^\pi \sin\theta_0\,\sigma_{11}(x_0)N_1(\theta,\theta_0)\,d\theta_0 = -\pi\{B_1 + C_1\cos\theta + D_1\cos 2\theta\},\quad 0\leqslant\theta\leqslant\pi \qquad ...(3\cdot23)$$

$$\int_0^\pi \sin\theta_0\,\sigma_{21}(x_0)N_2(\theta,\theta_0)\,d\theta_0 = -\pi\{B_2 + C_2\cos\theta + D_2\cos 2\theta\},\quad 0\leqslant\theta\leqslant\pi \qquad ...(3\cdot24)$$

where the constants involved are defined by

$$B_1 = \left[\frac{\pi^2\beta^2}{12} + \frac{1}{8}\left\{(1-b)^2 + \frac{1}{4}(1+b)^2\right\}\right]A_1,$$

$$B_2 = \left[\frac{\pi^2\beta^2}{6} - \frac{1}{8}\left\{(1-b)^2 + \frac{1}{4}(1+b)^2\right\}\right]A_2,$$

$$C_1 = -\frac{(1-b^2)}{8}A_1,\qquad D_1 = \frac{(1+b)^2}{64}A_1,$$

$$C_2 = \frac{(1-b^2)}{8}A_2,\qquad D_2 = -\frac{(1+b)^2}{64}A_2.$$

Eqns. (3·23) and (3·24) readily provide

$$\sin\theta_0\,\sigma_{11}(x_0) = -\left(\frac{B_1}{\lambda_1}\right) + C_1\cos\theta_0 + 2D_1\cos(2\theta_0) \qquad \text{...(3·25)}$$

$$\sin\theta_0\,\sigma_{21}(x_0) = \left(\frac{B_2}{\lambda_2}\right) + C_2\cos\theta_0 + 2D_2\cos(2\theta_0). \qquad \text{...(3·26)}$$

From (3·18), we have

$$\sin\theta_0 = 2\sqrt{(b-x_0)(1+x_0)}/(1+b). \qquad \text{...(3·27)}$$

From (3·11), (3·22), (3·25) and (3·27), we obtain the expression for the unknown charge density $\sigma_1(x_0)$ as

$$\sigma_1(x_0) = \sigma(x_0, -d) + \sigma(-x_0, d)$$

$$= \frac{(1+b)}{2\sqrt{(b-x_0)(1+x_0)}}\left[A_1 + \frac{1}{d^2}\left\{-\frac{B_1}{\lambda_1} + \frac{(1+2x_0-b)}{1+b}C_1 + 4\left[\left(\frac{1+2x_0-b}{1+b}\right)^2 - \frac{1}{2}\right]D_1\right\} + O\left(\frac{1}{d^4}\right)\right]. \qquad \text{...(3·28)}$$

From (3·12), (3·22), (3·26) and (3·27), we obtain the expression for the unknown charge density $\sigma_2(x_0)$ as

$$\sigma_2(x_0) = \sigma(x_0, -d) - \sigma(-x_0, d)$$

$$= \frac{(1+b)}{2\sqrt{(b-x_0)(1+x_0)}}\left[A_2 + \frac{1}{d^2}\left\{\frac{B_2}{\lambda_2} + \frac{(1+2x_0-b)}{1+b}C_2 + 4\left[\left(\frac{1+x_0-b}{1+b}\right)^2 - \frac{1}{2}\right]D_2\right\} + O\left(\frac{1}{d^4}\right)\right]. \qquad \text{...(3·29)}$$

Finally, we obtain expressions for the total charges Q_j, $j = 1, 2$ per unit length on the two strips

$$Q_1 = \int_{-1}^{b}\sigma(x_0, -d)\,dx_0 = \tfrac{1}{2}\int_{-1}^{b}[\sigma_1(x_0) + \sigma_2(x_0)]\,dx_0$$

$$= \left(\frac{1+b}{4}\right)\int_0^{\pi}\sin\theta_0\,[\sigma_1(x_0) + \sigma_2(x_0)]\,d\theta_0$$

$$= \frac{\pi(1+b)}{4}\left[(A_1 + A_2) + \frac{1}{d^2}\left\{-\frac{B_1}{\lambda_1} + \frac{B_2}{\lambda_2}\right\} + O\left(\frac{1}{d^4}\right)\right], \qquad \text{...(3·30)}$$

$$Q_2 = \left(\frac{1+b}{4}\right)\int_0^{\pi}\sin\theta_0\,[\sigma_1(x_0) - \sigma_2(x_0)]\,d\theta_0$$

$$= \frac{\pi(1+b)}{4}\left[(A_1 - A_2) - \frac{1}{d^2}\left\{\frac{B_1}{\lambda_1} + \frac{B_2}{\lambda_2}\right\} + O\left(\frac{1}{d^4}\right)\right], \qquad \text{...(3·31)}$$

which seem to be new results.

When $f \to \infty$ in (3·30) and (3·31), we get the values of total charges per unit length for the two equal parallel non-coplanar conducting strips, when these are charged to constant potentials V_1 and V_2 in a free space, and in this case, since

$$\beta \to 0, \quad A_1 = 0, \quad A_2 = 4(V_1 - V_2)/(1 + b)\lambda_2$$

we readily obtain

$$Q_1 = \frac{\pi(V_1 - V_2)}{\lambda_2}\left[1 - \frac{1}{8\lambda_2 d^2}\left\{(1-b)^2 + \frac{(1+b)^2}{4}\right\} + O\left(\frac{1}{d^4}\right)\right],$$

$$Q_2 = -Q_1 \qquad \text{...(3·32)}$$

Therefore, capacity C (in physical units) per unit length of the condenser formed, by the two equal parallel non-coplanar conducting strips is

$$C = \frac{Q_1 a_1}{V_1 - V_2} = \frac{\pi a_1}{\lambda_2}\left[1 - \frac{1}{8\lambda_2 d^2}\left\{(1-b)^2 + \frac{(1+b)^2}{4}\right\} + O\left(\frac{1}{d^4}\right)\right]. \qquad \text{...(3·33)}$$

When $b_1 \to a_1$, i.e., $b \to 1$, $\lambda_2 \to \log 4d$ in (3.33), we get the formula of capacity C_1 (in physical units) per unit length of the condenser, when the two parallel strips are lying exactly opposite to each other and the result is

$$C_1 = \frac{\pi a_1}{\log 4d}\left\{1 - \frac{1}{8d^2 \log 4d} + O\left(\frac{1}{d^4}\right)\right\}. \qquad \text{...(3·34)}$$

The above limiting formulae agree with the results given by Goel and Jain (1977) which serve as a check on our analysis.

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