ON CONSISTENCY OF T-MATRICES FOR (C, r)-BOUNDED SEQUENCES

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In this paper, the following main results have been proved besides lemmas. If T-matrices A and B satisfy the conditions (3.1) and (3.2) respectively, then A and B are consistent for $(C, 1)$-bounded sequences, provided any of the following conditions is satisfied :

(i) The matrices $F = (k \triangle a_{n,k})$ and $H = (k \triangle b_{n,k})$ commute.

(ii) FH and HF are absolutely equivalent for all bounded sequences.

(iii) $H^* \cap S \supset F^* \cap S$, where S is the set of the all-bounded sequences.

(iv) (z_k) is within the ranges of $F[H]$ and $H[F]$, where F and H are commutable and the $(C, 1)$-bounded sequence (s_k) is such that

$$z_k = (s_1 + s_2 + s_3 + \dots + s_k)/k.$$

The above results are extended for (C, r)-bounded sequences, where r is a positive integer.

1. INTRODUCTION

Appropriate conditions are known in order that T-matrices are consistent for bounded sequences (Cooke 1950, pp. 98; Diences 1957, pp. 412). Commutable row-finite T-matrices are consistent also for unbounded sequences (Cooke 1950, pp. 98; Diences 1957, pp. 312). In the present note, we give a set of appropriate conditions in order that T-matrices are consistent for $(C, 1)$-bounded sequences. We also extend these results for (C, r)-bounded sequences.

2. NOTATIONS, DEFINITIONS AND LEMMAS

$A(s_n)$ and A-lim s_n denote $\sum_{k=1}^{\infty} a_{n,k} s_k$ and $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} s_k$ respectively. A^* denotes the set of all sequences $\{s_n\}$ which are summable–A.

If A-lim $s_n = B$-lim s_n for $\{s_n\} \in A^* \cap B^*$ and the limit is finite, A and B are said to be consistent (Cooke 1950, pp. 96).

Lemma 1—If $\sum_{k=1}^{\infty} k \,|\Delta a_{n,k}| \leqslant M$ for every n and $\lim_{k\to\infty} a_{n,k} = 0$ for every n, then $\lim_{k\to\infty} k\, a_{n,k} = 0$ for every n where $\Delta a_{n,k} = a_{n,k} - a_{n,k+1}$ (Lemma 7 of Bosanquet 1945; also Cooke 1950, pp. 216, 218).

Lemma 2—If a T-matrix A satisfies the condition

$$\sum_{k=1}^{\infty} k \,|a_{n,k} - a_{n,k+1}| \leqslant M \text{ for every } n, \qquad \text{...(2.1)}$$

then the matrix $F = (f_{n,k})$ is a T-matrix, where $f_{n,k} = (k + r)(a_{n,k} - a_{n,k+1})$, $r = 0, 1, 2, 3, \cdots, p$ (p being some fixed integer).

PROOF: Since A is a T-matrix,

$$\lim_{k\to\infty} a_{n,k} = 0 \text{ for every } n. \qquad \text{...(2.2)}$$

Lemmas 1, (2.1) and (2.2) together imply that

$$\lim_{k\to\infty} k\, a_{n,k} = 0 \text{ for every } n. \qquad \text{...(2.3)}$$

Also, we have

$$\sum_{k=1}^{p} f_{n,k} = \sum_{k=1}^{p} (k \pm r)(a_{n,k} - a_{n,k+1})$$

$$= \sum_{k=1}^{p} k (a_{n,k} - a_{n,k+1}) \pm r \sum_{k=1}^{p} (a_{n,k} - a_{n,k+1})$$

$$= \sum_{k=1}^{p} a_{n,k} - p a_{n,p+1} \pm r (a_{n,1} - a_{n,p+1}) \qquad \text{...(2.4)}$$

Letting $p \to \infty$ in (2.4) and using (2.2) and (2.3), we get

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} f_{n,k} = 1 \qquad \text{...(2.5)}$$

Since A is a T-matrices, it follows from (2.1) that

$$\sum_{k=1}^{\infty} |f_{n,k}| \leqslant M_1 \text{ for every } n. \qquad \text{...(2.6)}$$

It is obvious that

$$\lim_{n\to\infty} f_{n,k} = 0 \text{ for every fixed } k. \qquad \text{...(2.7)}$$

Hence, F is a T-matrix, since F satisfies the conditions (2.5), (2.6) and (2.7).

Lemma 3—If the T-matrices A and B commute, A and B are consistent at least for bounded sequences (Cooke 1950, pp. 98).

Lemma 4—If A and B are T-matrices such that AB and BA are absolutely equivalent for all bounded sequences, then the corresponding A and B limits are consistent for all bounded sequences (Cooke 1950, pp. 132, eqns. 11).

Lemma 5—If every bounded sequence which is summable by a T-matrix A is also summable by another T-matric B, then A and B are consistent (Brudno 1945; Cooke 1950, pp. 130).

Lemma 6—If $\{s_k\}$ is within the ranges of A $[B]$ and B $[A]$, where A and B are commutable T-matrices, then A and B limits of $\{s_k\}$ are consistent (Cooke 1950, pp. 100).

We use the notation $A[B]$ to denote the transformation B followed by the transformation A and (AB) to denote the single transformation by the product matrix AB.

3. Consistency for $(C, 1)$-bounded Sequences

Theorem I—If the T-matrices A and B satisfy the conditions

$$\sum_{k=1}^{\infty} k \mid a_{n,k} - a_{n,k+1} \mid \leqslant M, \text{ for every } n \qquad \ldots(3\cdot 1)$$

and

$$\sum_{k=1}^{\infty} k \mid b_{n,k} - b_{n,k+1} \mid \leqslant M_2 \text{ for every } n \qquad \ldots(3\cdot 2)$$

respectively and the matrices $F = (k\Delta a_{n,k})$ and $H = (k\Delta b_{n,k})$ commute, then A and B are consistent for $(C, 1)$-bounded sequences.

Proof: Put

$$z_k = (s_1 + s_2 + s_3 + \cdots + s_k)/k \qquad \ldots(3\cdot 3)$$

then

$$s_k = kz_k - (k-1)\, z_{k-1} \; (k \geqslant 1), z_0 = 0. \qquad \ldots(3\cdot 4)$$

It is clear that if $\{s_k\}$ is any $(C, 1)$-bounded sequences, $\{z_k\}$ defined by (3·3) is bounded. Also,

$$\begin{aligned} \sum_{k=1}^{p} a_{n,k}\, s_k &= \sum_{k=1}^{p} a_{n,k} \{kz_k - (k-1)\, z_{k-1}\} \\ &= \sum_{k=1}^{p} k\; (a_{n,k} - a_{n,k+1})\; z_k + p\; a_{n,p+1}\, z_p \\ &= \sum_{k=1}^{p} f_{n,k}\, z_k + \; pa_{n,p+1}\; z_p \qquad \ldots(3\cdot 5) \end{aligned}$$

where

$$f_{n,k} = k\,(a_{n,k} - a_{n,k+1}) = k\Delta a_{n,k}.$$

Since A is a T-matrix,

$$\lim_{k\to\infty} a_{n,k} = 0 \text{ for every } n. \qquad \ldots(3\cdot 6)$$

Now, it follows from (3.1), (3.6) and Lemma 1 that

$$\lim_{k\to\infty} ka_{n,k} = 0 \text{ for every } n. \qquad ...(3.7)$$

But (3.6) and (3.7) together imply that

$$\lim_{k\to\infty} ka_{n,k+1} = 0 \text{ for every } n. \qquad ...(3.8)$$

Letting $p \to \infty$ in (3.5) and using (3.8) we get

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} s_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} f_{n,k} z_k. \qquad ...(3.9)$$

Similarly,

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} b_{n,k} s_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} h_{n,k} z_k \qquad ...(3.10)$$

where

$$h_{n,k} = k\Delta b_{n,k}.$$

The existence of the left-hand sides of (3.9) and (3.10) implies the existence of right-hand sides of (3.9) and (3.10) respectively.

If the T-matrices A and B satisfy the conditions (3.1) and (3.2) respectively, then it follows from Lemma 1, with $r = 0$, that $F = (k\Delta a_{n,k})$ and $H = (k\Delta b_{n,k})$ are T-matrices.

Since F and H are commutable, it follows from Lemma 3 that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} f_{n,k} z_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} h_{n,k} z_k. \qquad ...(3.11)$$

From (3.9), (3.10) and (3.11), we get

$$A - \lim s_n = B - \lim s_n.$$

i.e., A and B are consistent for $(C, 1)$-bounded sequences.

Theorem II—If the T-matrices A and B satisfy the conditions (3.1) and (3.2) respectively and FH and HF are absolutely equivalent for all bounded sequences where $F = (k\Delta a_{n,k})$ and $H = (k\Delta b_{n,k})$ then A and B are consistent for $(C, 1)$-bounded sequences.

PROOF: Let $\{s_k\}$ be any $(C, 1)$-bounded sequence, then the sequences $\{z_k\}$, defined by (3.3), is bounded. Now, we proceed as in Theorem I and consequently we get

$$A - \lim s_n = F - \lim z_n, \quad \text{where} \quad f_{n,k} = k\Delta a_{n,k} \qquad ...(3.12)$$

and

$$B - \lim s_n = H - \lim z_n, \quad \text{where} \quad h_{n,k} = k\Delta b_{n,k}. \qquad ...(3.13)$$

The existence of left-hand sides of (3·12) and (3·13) implies the existence of right-hand sides of (3·12) and (3·13) respectively. Since the *T*-matrices A and B satisfy conditions (3·1) and (3·2) respectively, it follows from Lemma 2 that F and H are *T*-matrices. Thus, it follows from Lemma 4 that the corresponding F and H limits are consistent for all bounded sequences $\{z_k\}$, since FH and HF are absolutely equivalent for all bounded sequences $\{z_k\}$. Therefore,

$$F - \lim z_n = H - \lim z_n . \qquad ...(3\cdot 14)$$

From (3·12), (3·13) and (3·14), we get

$$A - \lim s_n = B - \lim s_n .$$

Thus, A and B are consistent for $(C, 1)$-bounded sequences.

Theorem III—If the *T*-matrices A and B satisfy the conditions (3·1) and (3·2) respectively and $H^* \cap S \supset F^* \cap S$, where $F = (k\Delta a_{n,k})$, $H = (k\Delta b_{n,k})$ and S is the set of all bounded sequences, then A and B are consistent for $(C, 1)$-bounded sequences.

PROOF: Let $\{s_k\}$ be a $(C, 1)$-bounded sequence, then the sequence $\{z_k\}$, defined by (3.3), is bounded. Since the *T*-matrices A and B satisfy conditions (3·1) and (3·2) respectively, we proceed as in Theorem I and thus we get (3·9) and (3·10), where $F = (k\Delta a_{n,k})$ and $H = (k\Delta b_{n,k})$.

The existence of the left-hand sides of (3·9) and (3·10) implies the existence of their right-hand sides respectively.

The *T*-matrices A and B satisfy the conditions (3·1) and (3·2) respectively. Then (from Lemma 2) F and H are *T*-matrices.

Now, it follows from Lemma 5 that

$$F - \lim z_n = H - \lim z_n \qquad ...(3\cdot 15)$$

since F and H are *T*-matrices, $H^* \cap s \supset F^* \cap S$ and $\{z_k\}$ is bounded. From (3·9), (3·10) and (3·15), we get

$$A - \lim s_n = B - \lim s_n.$$

Thus, A and B are consistent for $(C, 1)$-bounded sequences.

Theorem IV—If the *T*-matrices A and B satisfy the conditions (3·1) and (3·2) and $\{z_k\}$ is within the ranges of $F[H]$ and $H[F]$, where $F = (k\Delta a_{n,k})$ and $H = (k\Delta b_{n,k})$ are commutable, then A and B are consistent for $(C, 1)$-bounded sequences $\{s_k\}$ whenever $z_k = (s_1 + s_2 + s_3 + \cdots + s_k)/k$.

PROOF: Let $z_k = (s_1 + s_2 + s_3 + \cdots + s_k)/k$, where $\{s_k\}$ is $(C, 1)$-bounded sequence. Then $\{z_k\}$ is bounded and

$$s_k = k\, z_k - (k-1)\, z_{k-1}\, (k \geqslant 1).$$

Since the *T*-matrices A and B satisfy conditions (3·1) and (3·2), we proceed as in Theorem I and finally we get (3·9) and (3·10). The existence of the left-hand sides of (3·9) and (3·10) implies the existence of their right-hand sides respectively.

It follows from Lemma 2 that $F = (k\Delta a_{n,k})$ and $H = (k\Delta b_{n,k})$ are T-matrices and $\{z_k\}$ is within the ranges of $F[H]$ and $H[F]$. Then (3·15) holds (from Lemma 6). Hence

$$A - \lim s_n = B - \lim s_n$$

i.e., A and B are consistent for $(C, 1)$-bounded sequences.

4. Notations and Lemmas

We write

$$\Delta_1 [a_{n,k}] = k\Delta a_{n,k} = k (a_{n,k} - a_{n,k+1}),$$

$$\Delta_2 [a_{n,k}] = \Delta_1 [\Delta_1 (a_{n,k})] = k\Delta (k\Delta a_{n,k}),$$

$$\Delta_3 [a_{n,k}] = \Delta_2 [\Delta_1 [a_{n,k}]] = k\Delta \big(k\Delta (k\Delta a_{n,k})\big),$$

$$\ldots = \ldots = \ldots$$

$$\Delta_r [a_{n,k}] = \Delta_{r-1} [\Delta_1 (a_{n,k})], \text{ i.e., } \Delta_r [a_{n,k}]$$

denotes the operations of $k\Delta$ on $a_{n,k}$ successively r times.

It may be noted that $k^r \Delta^r a_{n,k} \neq \Delta_r [a_{n,k}]$. For example, taking $r = 2$,

$$\Delta_2 [a_{n,k}] = k^2 \Delta^2 a_{n,k} - k\Delta a_{n,k+1}$$

which is not the same as $k^2 \Delta^2 a_{n,k}$.

Lemma 7 If A is a T-matrix satisfying the condition

$$\sum_{k=1}^{\infty} k \,|\, \Delta a_{n,k} \,| \leqslant M \text{ for every } n,$$

then $(f_{n,k})$ is a T-matrix, where $f_{n,k} = k\Delta a_{n,k}$.

The Lemma follows from Lemma 2, with $r = 0$.

Lemma 8—If $f_{n,k} = k (a_{n,k} - a_{n,k+1})$ and $h_{n,k} = k (b_{n,k} - b_{n,k+1})$, then the commutability of the matrices $(\Delta_r [a_{n,k}])$ and $(\Delta_r [b_{n,k}])$ implies the commutability of $(\Delta_{r-1} [f_{n,k}])$ and $(\Delta_{r-1} [h_{n,k}])$.

Proof: We have

$$\Delta_{r-1} [f_{n,k}] = \Delta_{r-1} [\Delta_1 [a_{n,k}]] = \Delta_r [a_{n,k}], \qquad \ldots(4\cdot 1)$$

and

$$\Delta_{r-1} [h_{n,k}] = \Delta_{r-1} [\Delta_1 [b_{n,k}]] = \Delta_r [b_{n,k}]. \qquad \ldots(4\cdot 2)$$

From (4·1), (4·2) and by the hypothesis, $(\Delta_{r-1} [f_{n,k}])$ and $(\Delta_{r-1} [h_{n,k}])$ are commutable.

Lemma 9—If $f_{n,k} = k (a_{n,k} - a_{n,k+1})$ and $h_{n,k} = k (b_{n,k} - b_{n,k+1})$, then the absolute equivalence of the matrices $(\Delta_r [a_{n,k}])$ and $(\Delta_r [b_{n,k}])$ for all bounded sequences implies the absolute equivalence of the matrices $(\Delta_{r-1}[f_{n,k}])$ and $(\Delta_{r-1}[h_{n,k}])$ for all bounded sequences.

The lemma follows immediately from (4·1), (4·2) and by the hypothesis.

Lemma 10—If $f_{n,k} = k\Delta a_{n,k}$ and $h_{n,k} = k\Delta b_{n,k}$, then $(\Delta_r[b_{n,k}])^* \cap S \supset (\Delta_r[a_{n,k}])^* \cap S$ implies that $(\Delta_{r-1}[h_{n,k}])^* \cap S \supset (\Delta_{r-1}[f_{n,k}])^* \cap S$, where S is the set of all bounded sequences.

PROOF: From (4·1) and (4·2), we get

$$(\Delta_{r-1}[h_{n,k}])^* \cap S = (\Delta_r[b_{n,k}])^* \cap S$$

and

$$(\Delta_{r-1}[f_{n,k}])^* \cap S = (\Delta_r[a_{n,k}])^* \cap S.$$

Therefore, by the hypothesis,

$$(\Delta_{r-1}[h_{n,k}])^* \cap S \supset (\Delta_{r-1}[f_{n,k}])^* \cap S.$$

Lemma 11—If $f_{n,k} = k\Delta a_{n,k}$ and $h_{n,k} = k\Delta b_{n,k}$, then $\{z_k\}$ is within the ranges of $(\Delta_r[a_{n,k}])$ $[(\Delta_r[b_{n,k}])]$ and $(\Delta_r[b_{n,k}])$ $[(\Delta_r[a_{n,k}])]$ implies that $\{z_k\}$ is within the ranges of $(\Delta_{r-1}[f_{n,k}])[(\Delta_{r-1}[h_{n,k}])]$ and $(\Delta_{r-1}[h_{n,k}])$ $[(\Delta_{r-1}[f_{n,k}])]$.

PROOF: From (4·1) and (4·2), we get

$$(\Delta_{r-1}[f_{n,k}])[(\Delta_{r-1}[h_{n,k}])] = (\Delta_r[a_{n,k}])[(\Delta_r[b_{n,k}])]$$

and

$$(\Delta_{r-1}[h_{n,k}])[(\Delta_{r-1}[f_{n,k}])] = (\Delta_r[b_{n,k}])[(\Delta_r[a_{n,k}])].$$

Hence, the lemma follows.

Lemma 12—If r is a positive integer, then $\{s_k\}$ is (C, r)-bounded if and only if $\{z_n\}$ is $(C, {}_{r-1})$-bounded, where

$$z_n = (s_1 + s_2 + s_3 + \cdots s_n)/n.$$

This is essentially Lemma 1, of Bosanquet (1945).

Lemma 13—If $\{s_n\}$ is (C, r)-bounded where r is a positive integer, then $s_n = O(n^r)$ (Jha 1962, pp. 117; Bosanquet 1945, Lemma 3).

Lemma 14—If $f_{n,k} = k\ (a_{n,k} - a_{n,k+1})$, then $\sum_{k=1}^{\infty} k^J \,|\, \Delta^J a_{n,k} \,| \leqslant M$ for every n $(J = r, r-1)$ implies that $\sum_{k=1}^{\infty} k^{r-1} \,|\, \Delta^{r-1} f_{n,k} \,| \leqslant M$, for every n, where r is a positive integer (Zaman 1972, pp. 105).

5. CONSISTENCY FOR (C, r)-BOUNDED SEQUENCES

In this section, we extend the results of section 3 for (C, r)-bounded sequences, where r is a positive integer.

Theorem V—If the T-matrices A and B satisfy the conditions

$$\left.\begin{array}{l}\text{(i) } \sum_{k=1}^{\infty} k^j \,|\, \Delta^j a_{n,k} \,| \leqslant M_1 \text{ for every } n, \; (j = 1, 2, 3, \cdots, r) \\ \text{(ii) } \lim_{k\to\infty} k^r a_{n,k} = 0 \text{ for every } n \end{array}\right\} \qquad \ldots(5{\cdot}1)$$

and

$$\left.\begin{array}{l}\text{(i) } \sum_{k=1}^{\infty} k^j \,|\, \Delta^j b_{n,k} \,| \leqslant M_2 \text{ for every } n, \; (j = 1, 2, 3, \cdots, r) \\ \text{(ii) } \lim_{k\to\infty} k^r b_{n,k} = 0 \text{ for every } n \end{array}\right\} \qquad \ldots(5{\cdot}2)$$

respectively and the matrices $(\Delta_j [a_{n,k}])$ and $(\Delta_j [b_{n,k}])$, $(j = 1, 2, 3, \cdots, r)$, are commutable, then A and B are consistent for (C, r)-bounded sequences, where r is a positive integer.

PROOF: Putting in $r = 1$ in (5·1) and (5·2), we proceed as in Theorem I and consequently the theorem holds for $r = 1$. Suppose the theorem holds for $r = t - 1$, then we shall prove that the theorem is also true for $r = t$.

Put

$$z_k = (s_1 + s_2 + s_3 + \cdots + s_k)/k, \qquad \ldots(5{\cdot}3)$$

so that,

$$s_k = k\, z_k - (k - 1)\, z_{k-1} \, (k \geqslant 1). \qquad \ldots(5{\cdot}4)$$

Also, we have

$$\sum_{k=1}^{p} a_{n,k}\, s_k = \sum_{k=1}^{p} a_{n,k} \{k z_k - (k - 1)\, z_{k-1}\}$$

$$= \sum_{k=1}^{p} k\, (a_{n,k} - a_{n,k+1})\, z_k + p\, a_{n,p+1}\, z_p$$

$$= \sum_{k=1}^{p} f_{n,k}\, z_k + p\, a_{n,p+1}\, z_p, \qquad \ldots(5{\cdot}5)$$

where

$$f_{n,k} = k\; (a_{n,k} - a_{n,k+1}).$$

Similarly,

$$\sum_{k=1}^{p} b_{n,k}\, s_k = \sum_{k=1}^{p} h_{n,k}\, z_k + p\, b_{n,p+1}\, z_p, \qquad \ldots(5{\cdot}6)$$

where

$$h_{n,k} = k\; (\Delta\, b_{n,k+1}).$$

It follows from Lemma 12 that the sequence $\{s_k\}$, defined by (5·3), is (C, t)-bounded if and only if $\{z_k\}$ is $(C, t - 1)$-bounded. This also follows from Lemma 13 $z_k = O(k^{t-1})$.

The condition (5.1) (ii) implies that

$$\lim_{p\to\infty} p\, a_{n,p+1} z_p = 0, \quad \text{when} \quad z_p = O(p^{t-1}) \qquad \dots(5.7)$$

From (5.5) and (5.7), we get

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} s_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} f_{n,k} z_k. \qquad \dots(5.8)$$

Similarly,

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} b_{n,k} s_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} h_{n,k} z_k. \qquad \dots(5.9)$$

By Lemma 7, the conditions (5.1) (i) and (5.2) (i), with $r = 1$, imply respectively that $(f_{n,k})$ and $(h_{n,k})$ are T-matrices.

The conditions (5.1) (ii) and (5.2) (ii) imply respectively that

$$\lim_{k\to\infty} k^{t-1} f_{n,k} = 0 \text{ for every } n, \qquad \dots(5.10)$$

and

$$\lim_{k\to\infty} k^{t-1} h_{n,k} = 0 \text{ for every } n. \qquad \dots(5.11)$$

By Lemma 14, with $r = t$, condition (5.1) (i) implies that

$$\sum_{k=1}^{\infty} k^{t-1} \left| \Delta^{t-1} f_{n,k} \right| \leqslant M_3 \text{ for every } n \qquad \dots(5.12)$$

Similarly, (5.2) (i) implies that

$$\sum_{k=1}^{\infty} k^{t-1} \left| \Delta^{t-1} h_{n,k} \right| \leqslant M_4 \text{ for every } n. \qquad \dots(5.13)$$

Now the matrices $(f_{n,k})$ and $(h_{n,k})$ satisfy the conditions (5.10) and (5.12) and (5.11) and (5.13) respectively and also the matrices $(\Delta_{t-1} [f_{n,k}])$ and $(\Delta_{t-1} [h_{n,k}]$ are commutable (from Lemma 8, with $r = t$). Thus, the T-matrices $(f_{n,k})$ and $(h_{n,k})$ are consistent for $(C, t-1)$-bounded sequences $\{z_k\}$ whenever they exist (for by the supposition, the theorem holds for $r = t-1$). Therefore,

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} f_{n,k} z_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} h_{n,k} z_k, \qquad \dots(5.14)$$

by virtue of the observation made above.

From (5.8), (5.9) and (5.14), we get

$$A - \lim s_n = B - \lim s_n.$$

Hence, A and B are consistent for (C, t)-bounded sequences.

Theorem VI—If the T-matrices A and B satisfy the conditions (5.1) and (5.2) respectively and the matrices

$$(\Delta_j [a_{n,k}]) (\Delta_j [b_{n,k}]) \quad \text{and} \quad (\Delta_j [b_{n,k}]) (\Delta_j [a_{n,k}]), \quad (j = 1, 2, 3, \dots r)$$

are absolutely equivalent for all bounded sequences, then A and B are consistent for (C, r) bounded sequences, where r is a positive integer.

PROOF: Putting $r = 1$ in (5·1) and (5·2), we proceed as in Theorem II. Then the theorem is true for $r = 1$. Suppose that the theorem is true for $r = t - 1$, then we shall prove that the theorem is also true for $r = 1$.

Since A and B are T-matrices satisfying the conditions (5·1) and (5·2) respectively we proceed as in Theorem V so that (5·8) and (5·9) hold.

We have proved in Theorem V that $(f_{n,k})$ and $(h_{n,k})$ are T-matrices (from Lemma 7) satisfying the conditions (5·10) and (5·12) & (5·11) and (5·13) respectively. Moreover, $(\Delta_{t-1} [f_{n,k}])$ $(\Delta_{t-1} [h_{n,k}])$ and $(\Delta_{t-1} [h_{n,k}])$ $(\Delta_{t-1} [f_{n,k}])$ are absolutely equivalent for all bounded sequences (from Lemma 9, with $r = t$). Thus, $(f_{n,k})$ and $(h_{n,k})$ are consistent for $(C, t - 1)$-bounded sequences $\{z_k\}$, since by the supposition, the theorem is true for $(r = t - 1)$.

By virtue of the observations made above, (5·14) holds. Now it follows from (5·8), (5·9) and (5·14) that $A - \lim s_n = B - \lim s_n$.

Hence, A and B are consistent for (C, t)-bounded sequences.

Theorem VII—If the T-matrices A and B satisfy the conditions (5·1) and (5·2) respectively and

$$(\Delta_j [b_{n,k}])^* \cap S \supset (\Delta_j \, a_{n,k})^* \cap S \; (j = 1, 2, 3 \dots r),$$

where S is the set of all bounded sequences and r is a positive integer, then A and B are consistent for (C, r)-bounded sequences.

PROOF: Putting $r = 1$ in (5·1) and (5·2), it follows from Theorem III that the theorem holds for $r = 1$. Let the theorem hold for $r = t - 1$; then we shall prove that the theorem holds also for $r = t$.

The T-matrices A and B satisfy the conditions (5·1) and (5·2) respectively and we proceed as in Theorem V, so that (5·8) and (5·9) hold.

We have also proved in Theorem V that the T-matrices $(f_{n,k})$ and $(h_{n,k})$ satisfy the conditions (5·10) and (5·12) & (5·11) and (5·13) respectively. By Lemma 10, with $r = t$, we have

$$(\Delta_{t-1} [h_{n,k}])^* \cap S \supset (\Delta_{t-1} [f_{n,k}])^* \cap S.$$

Since the theorem holds for $r = t - 1$, the T-matrices $(f_{n,k})$ and $(h_{n,k})$ are consistent for $(C, t - 1)$-bounded sequence $\{z_k\}$.

Now (5·14) holds by virtue of the observations made above. Thus, it follows from (5·8), (5·9) and (5·14) that $A - \lim s_n = B - \lim s_n$. This completes the proof of the theorem.

Theorem VIII—If the T-matrices A and B satisfy the conditions (5·1) and (5.2) respectively and $\{z_k\}$ is within the ranges of

$$(\Delta_j [a_{n,k}]) \, [(\Delta_j [b_{n,k}])] \quad \text{and} \quad (\Delta_j [b_{n,k}]) \, [(\Delta_j [a_{n,k}])] \; (j = 1, 2, 3 \dots r),$$

where $(\Delta_j [a_{n,k}])$ and $(\Delta_j [b_{n,k}])$ $(j = 1, 2, 3, \cdots r)$ are commutable and r being a positive integer, then A and B are consistent for (C, r)-bounded sequences $\{s_k\}$ whenever $z_k = (s_1 + s_2 + s_3 + \cdots + s_k)/k$.

PROOF: Putting $r = 1$, in (5·1) and (5·2), it follows from Theorem IV that the theorem is true for $r = 1$. If we suppose that the theorem is true for $r = t-1$, we shall prove that the theorem is also true for $r = t$.

It is clear that $\{z_k\}$ is $(C, t-1)$-bounded sequence (from Lemma 12). We have proved in Theorem V that the T-matrices $(f_{n,k})$ and $(h_{n,k})$ satisfy the conditions (5·10) and (5·12) & (5·11) and (5·13) respectively. Also, (5·8) and (5.9) hold.

By Lemma 11, with $r = t$, z_k is within the ranges of

$$(\Delta_{t-1} [f_{n,k}]) [(\Delta_{t-1} [h_{n,k}])] \quad \text{and} \quad (\Delta_{t-1} [h_{n,k}]) [(\Delta_{t-1} [f_{n,k}])];$$

and $(\Delta_{t-1} [f_{n,k}])$ and $(\Delta_{t-1} [h_{n,k}])$ are commutable (from Lemma 8, with $r = t$). By supposition, the T-matrices $(f_{n,k})$ and $(h_{n,k})$ are consistent for $(C, t-1)$-bounded sequences and, therefore, (5·14) holds. Thus, (5·8), (5·9) and (5·14) together imply that $A - \lim s_n = B - \lim s_n$. Consequently, the theorem is true for (C, t)-bounded sequences.

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