APPLICATIONS OF THE PROJECTION THEOREM AND SOME RESULTS

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In this paper, we give applications of the projection theorem proved by the authors for uniformly convex Banach spaces. The method used is the semi-inner product technique. We obtain in the sequel generalizations of Hilbert space results to some Banach spaces.

1. INTRODUCTION

Pethe and Thakare (1975) proved the projection theorem in uniformly convex Banach spaces. The result in essence says that if X is a uniformly convex Banach space and M is any closed subspace of X, then any element x of X can be uniquely expressed as $(y+z)$, where y is in M and z in $M^{\perp}$. The purpose of this paper is to give some interesting applications of the projection theorem. The method used is the semi-inner product technique. The following definition is necessary for the development of the theory.

Definition—A semi-inner product (s.i.p. for short) on a complex vector space X is a complex function (x, y) on $(X, \times X)$ with the following properties :

(1) $(x + y, z) = (x, z) + (y, z)$
and $(\lambda x, y) = \lambda (x, y)$ for all complex λ

(2) $(x, x) > 0$ when $x \neq 0$

(3) $| (x, y) |^2 \leqslant (x, x) (y, y)$.

An s.i.p. is said to satisfy the homogeneity property, if $(x, \lambda y) = \bar{\lambda} (x, y)$ for all complex λ.

We note that every semi-inner product space is a normed linear space with $\| x \| = (x, x)^{1/2}$. Also, every normed vector space can be represented as an s.i.p. space with the homogeneity property.

For detailed discussion of these concepts, one can refer to Lumer (1961) or Giles (1967).

2. RESULTS TAKING X TO BE A UNIFORMLY CONVEX BANACH SPACE

In this article we shall take X to be a uniformly convex Banach space (over the complex field) on which there is a fixed s.i.p defined. Further, we assume that this s.i.p satisfies the homogeneity property. Result 2.1 states the conditions under which an operator on X is a projection operator. In Result 2.2, we discuss the n-fold transitive property of a compact operator on X. Both these results are generalizations of the results to be found in Berberian (1961). We further investigate projections in relation to subspaces in the context of X. In this connection, the work of Berkson (1972) may be referred to.

Result 2.1—Let N be a closed subspace of X. $T: X \to X$ is a mapping satisfying

(i) $(Tx_1, x_2) = (x_1, Tx_2)$ For all x_1, x_2 in X

(ii) $Ty = y$ for $y \in N$.

(iii) $Tz = 0$ for $Z \in N^{\perp}$

Then T is a projection operator on N.

Here (—, —) is any but fixed s.i.p. defined on X.

PROOF : Since X is uniformly convex, each x in X can be looked upon as $y + z$, where $y \in N$ and $z \in N^{\perp}$. To prove $T(x) = y$, we first show that T is additive. Consider $(T(y+b), w) = (y+b, T(w))$ by (i), and $(T(y) + T(b), w) = (T(y), w) + (T(b), w) = (y, T(w)) + (b, T(w) = (y+b, T(w))$,
So that $(T(y+b), w) = (T(y) + T(b), w)$ for all w in X
Hence,

$$T(y+b) = T(y) + T(b).$$

Now,

$$T(x) = T(y+z) = T(y) + T(z) = y.$$

Result 2.2—If y and z are fixed vectors in X, $y \neq 0$, there exists a compact operator on X such that $Ty = z$. More generally, the set of compact operators is n-fold transitive in the following sense :

If $y_1, y_2 \ldots y_n$ are linearly independent and $z_1, z_2 \ldots z_n$ are arbitrary, there exists a compact operator T such that $Ty_k = z_k$ for all k.

PROOF : Let N be n-dimensional linear subspace generated by y_k's, and define $T: N \to X$, such that $Ty_k = z_k$. Clearly T is continuous (see Berberian 1961, p. 80).

Define $T = 0$ on $N^{\perp}$; and extend T to the full space X *via* the relation that every vector in X can be expressed uniquely as sum of vectors in N and $N^{\perp}$. Thus, T is finite dimensional operator and hence completely continuous.

Result 2·3—Let E be a projection operator on X in which orthogonality is symmetric.

If x in X is such that $\| x \| = \| Ex \|$, then $x \in R(E)$.

PROOF : Let (., .) be a fixed but arbitrary s.i.p. an X. Since X is uniformly convex $x = y + z$, where $y \in R(E)$ and $z \perp y$.

Now $\| x \| = \| y + z \| = \| Ex \| = \| y \|$.

Also $(z, y) = 0$.

Since the space is strictly convex, using the argument of Torrance (1970), we get $z = 0$. Hence, $x = y \in R(E)$.

Result 2.4—Let $N = \{x_1, x_2 \ldots x_n\}$ be finite linearly independent unit vectors in X in which orthogonality is symmetric. For any vector x in X, the orthogonal projection of x into the subspace spanned by N is $y = \sum_{k=1}^{n} (x, x_k) x_k$, where (.,.) is any but fixed s.i.p. defined on X.

PROOF : Since x is in X and X is uniformly convex, $x = y + z$, where $y \in [N]$ and $z \in [N]^{\perp}$. Now

$$\sum_{k=1}^{n} (x, x_k) x_k = \sum_{k=1}^{n} (y + z, x_k) x_k$$

$$= \sum_{k=1}^{n} (y, x_k) x_k$$

$$= \sum_{k=1}^{n} (\alpha_1 x_1 + \ldots + \alpha_n x_n, x_k) x_k$$

$$= \sum_{k=1}^{n} \alpha_k (x_k, x_k) x_k = \sum_{k=1}^{n} \alpha_k x_k = y.$$

3. RESULTS TAKING X TO BE ANY BANACH SPACE WITH FIXED S.I.P.

In this section, we shall assume that X is *any* Banach space on which there is defined a fixed s.i.p. In Result 3.1, we generalize the result of Husain and Malavia (1972) wherein we establish that in any Banach space strong convergence is equivalent to uniform weak convergence. It is known that if M and N are orthogonal closed subspaces of a Hilbert space, then $M + N$ is closed. By dropping the assumption of orthogonality, we obtain in Result 3.2, an analogue of the result of Halmos (1967, problem 8, p. 7) for any Banach space.

Finally, we solve the problem raised by Winter (1972, advanced problem 5842) for any Banach space.

Result 3.1—In a Banach space, strong convergence is the same as weak convergence uniformly on the unit sphere more precisely: $\| x_n - x \| \to 0$ if, and only if $(x_n, y) \to (x, y)$ uniformly for $\| y \| = 1$.

PROOF : It is sufficient to treat the case $x = 0$. If $\| x_n \| \to 0$, then since $|(x_n, y)| \leqslant \| x_n \| \| y \| = \| x_n \|$ whenever $\| y \| = 1$, it follows that $(x_n, y) \to 0$ uniformly, as stated.

Suppose, conversely that for each $\varepsilon > 0$, if n is sufficiently large, then $|(x_n, y)| < \varepsilon$ whenever $\| y \| = 1$. From this, we conclude that the size of n that is needed does not depend on y. It follows that if n is sufficiently large, then $|(x_n, y/\| y \|)| < \varepsilon$ whenever $y \neq 0$ and hence $|(x_n, y)| < \varepsilon \| y \|$ for all y. Thus, in particular, if n is sufficiently large; then $\| x_n \|^2 \leqslant \varepsilon x_n$, (put $y = x_n$) or $\| x_n \| < \varepsilon$.

We observe here that the argument is perfectly general; it applies to all nets and not to sequences only.

Result 3.2—If M is a finite dimensional linear subspace of X and N is a closed linear subspace of X, then the vector sum $(M + N)$ is necessarily closed.

PROOF : It is sufficient to treat the case when $\dim M = 1$. The general case is obtained by induction on dimension. Suppose, therefore, that M is spanned by a single vector x_0, so that $M + N$ consists of all vectors of the form $\alpha x_0 + y$ where α is a scalar and y is in N. If $x_0 \in N$, then $M + N = N$, nothing to prove.

If $x_0 \notin N$, let y_0 be the projection of x_0 in N; i.e., y_0 is the unique vector in N for which $(x_0 - y_0) \perp N$. Now since y is in N

$$\| \alpha x_0 + y \|^2 = \| \alpha (x_0 - y_0) + (\alpha y_0 + y) \|^2$$

$$\geqslant |\alpha|^2 \| x_0 - y_0 \|^2 \; [\because (x_0 - y_0) \perp (\alpha y_0 + y)].$$

Or

$$|\alpha| \leqslant \frac{\| \alpha x_0 + y \|}{\| x_0 - y_0 \|}$$

and, therefore, $\| y \| = \| \alpha x_0 + y - \alpha x_0 \|$

$$\leqslant \| \alpha x_0 + y \| + \frac{\| \alpha x_0 + y \|}{\| x_0 - y_0 \|} \cdot \| x_0 \|.$$

These inequalities imply that $M + N$ is closed. Indeed, if $\alpha_n x_0 + y_n \to h$; so that $\{ \alpha_n x_0 + y_n \}$ is a Cauchy sequence, then the inequalities imply that both $\{\alpha_n\}$ and $\{y_n\}$ are Cauchy sequences. It follows that $\alpha_n \to \alpha$ and $y_n \to y$ say, with y in N, of course, and consequently $h = \lim_n (\alpha_n x_0 + y) = x_0 + y$.

Result 3.3—Let T be a linear map of X into itself. Suppose there exists a subset S such that $T(x) \in S$ and $(x - Tx) \in S^{\perp}$ for all x in X. Then S is a closed linear subspace. Furthermore, if T is symmetric, then T is an orthogonal projection of X onto S.

PROOF : If $x \in S$, since $Tx \in S$ and $(x - Tx) \in S^{\perp}$, we have $(x - Tx) \perp (x - Tx)$, so that $Tx = x$. We thus see that $x \in S$, iff $Tx = x$. Now to show that $T^2 = T$ on X. For any $y \in X$, $T(y) \in S$, so that

$$T^2(y) = T[T(y)] = T(y),$$

and thus T is a projection. Now if T is symmetric and if $x \in R(T)$ and $y \in N(T)$, then $T(x) = x$ and $T(y) = 0$, so that $(x, y) = (Tx, y) = (x, Ty) = (x, 0) = 0$, where $(.,.)$ is any s.i.p. compatible with the norm structure on X.

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