A FIXED POINT THEOREM IN BANACH SPACE

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(*Received* 14 *March* 1975 ; *after revision* 21 *July* 1976)

In this paper, a characterization of reflexivity due to Smulian (1939) has been used to prove a fixed point theorem in a reflexive Banach space. The theorem extends the recent theorems of Woodward (1975) and Kirk (1965).

INTRODUCTION

Let X denote a Banach space, and K, a bounded convex subset of X. We define the diameter of K by $D(K) = \sup \{\| x - y \| : x, y \in K\}$. K is said to have normal structure, if for each convex subset H of K, which contains more than one point, there is a point $x \in H$, such that

$$\text{Sup } \{\| x - y \| : y \in H\} < D(H)$$

Let T be a self-mapping of X. T is said to be nonexpansive if $\| Tx - Ty \| \leqslant \| x - y \| \; \forall \; x, y \in X$ By $O(x)$ we denote

$$O(x) = \{x, Tx, T^2x, \ldots\ldots\}$$

Finally, C_0K and $\overline{C_0K}$ will denote the convex hull and closed convex hull of K.

Smulian (1939) proved that a necessary and sufficient condition that a Banach space X is reflexive is that: (*a*) Every bounded descending sequence (transfinite) of nonempty closed convex subsets of X has a nonempty intersection. Kirk (1970) proved the following theorem :

Theorem A—Let K be a nonempty, bounded, closed and convex subset of a reflexive Banach space X, and suppose that K has a normal structure. If T is a nonexpansive mapping of K into itself, then T has a fixed point.

Recently, Woodward (1975) proved that the conclusion of Theorem A of Kirk (1965) remains true if $T : K \to K$ satisfies:

$$\| Tx - Ty \| \leqslant \tfrac{1}{2} \| x - Tx \| + \tfrac{1}{2} \| y - Ty \| \;\; \forall \; x, y \in K.$$

Further, T has a unique fixed point in this case.

AMS (MOS) subject classification (1970) Primary; 471170; Secondary; 54H25 Keywords and phrases; Fixed point, reflexive Banach space, normal structure, convex, non-expansive.

We shall prove the following theorem :

Theorem—Let K be a nonempty, bounded, closed and convex subset of a reflexive Banach space X and let K have a normal structure. If T be a mapping of K into itself, such that

$$\| Tx - Ty \| \leqslant \alpha \| x - y \| + \beta \{\| x - Tx \| + \| y - Ty \|\} + \gamma \{ \|x - Ty \| + \| y - Tx \|\}$$

for all $x, y \in K$ and for some $\alpha, \beta, \gamma \in R^+$ with $3\alpha + 2\beta + 4\gamma \leqslant 1$, then T has a unique fixed point.

Remarks— (1) The theorem of Woodward (1975) follows from our theorem by taking $\beta = \frac{1}{2}$, $\alpha = \gamma = 0$ and the theorem of Kirk (1965) follows by taking $\beta = \gamma = 0$ and $3\alpha = \lambda = 1$.

(2) It is not known whether normal structure is essential for the results of this paper (Kirk 1965).

Lemma A—Let K be a subset of a Banach space X and let the following condition

$T : K \to K$ be satisfied.

(B) $$\| Tx - Ty \| \leqslant \alpha \| x - y \| + \beta \{\| x - Tx \| + \| y - Ty \|\} + \gamma \{\|x - Ty \| + \| y - Tx \| \}$$

for all $x, y \in K$ and for some $\alpha_1, \beta_1, \gamma \in R^+$ with $3\alpha + 2\beta + 4\gamma \leqslant 1$.
Then, for $x \in K$, $\| T^n x - T^{n+1} x \| \leqslant \| T^{n-1} x - T^n x \|$ and $\| T^m x - T^n x \| \leqslant \| x - Tx \|$ for positive integers m and n.

Proof : With $y = Tx$ in (B), one gets

$$\begin{aligned}\| Tx - T^2x \| &\leqslant \alpha \| x - Tx \| + \beta \{\| x - Tx \| + \| Tx - T^2x \|\} + \gamma \{\| x - T^2x \| + \| Tx - Tx \|\} \\ &\leqslant (\alpha + \beta + \gamma) \| x - Tx \| + (\beta + \gamma) \| Tx - T^2x \|\end{aligned}$$

i.e.,

$$\begin{aligned}\| Tx - T^2x \| &\leqslant \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} \| x - Tx \| \\ &\leqslant \frac{1 - 2\alpha - \beta - 3\gamma}{1 - \beta - \gamma} \| x - Tx \| \\ &= \left[\frac{(1 - \beta - \gamma) - (2\alpha + 2\gamma)}{1 - \beta - \gamma)}\right] \| x - Tx \| \\ &= \left[1 - \frac{(2\alpha + 2\gamma)}{1 - \beta - \gamma}\right] \| x - Tx \| \leqslant \| x - Tx \|\end{aligned}$$

Replacing x by $T^{n-1} x$, one has

$$\| T^n x - T^{n+1} x \| \leqslant \| T^{n-1} x - T^n x \|$$

Now, $\| T^m x - T^n x \| \leqslant \alpha \| T^{m-1} x - T^{n-1} x \|$

$$+ \beta \{ \| T^{m-1} x - T^m x \| + \| T^{n-1} x - T^n x \| \}$$

$$+ \gamma \{ \| T^{m-1} x - T^n x \| + \| T^{n-1} x - T^m x \| \}$$

i.e.,

$$\begin{aligned} \| T^m x - T^n x \| &\leqslant \alpha \| T^{m-1} x - T^m x \| + \alpha \| T^m x - T^n x \| \\ &\quad + \alpha \| T^n x - T^{n-1} x \| + \beta \| T^{m-1} x - T^m x \| \\ &\quad + \beta \| T^{n-1} x - T^n x \| + \gamma \| T^{m-1} x - T^m x \| \\ &\quad + \gamma \| T^m x - T^n x \| + \gamma \| T^{n-1} x - T^n x \| \\ &\quad + \gamma \| T^n x - T^m x \| \\ &= (\alpha + \beta + \gamma) \| T^{m-1} x - T^m x \| + (\alpha + \beta + \gamma) \| T^{n-1} x - T^n x \| \\ &\quad + (\alpha + 2\gamma) \| T^m x - T^n x \|. \end{aligned}$$

Hence,

$$\begin{aligned} \| T^m x - T^n x \| &\leqslant \frac{\alpha + \beta + \gamma}{1 - \alpha - 2\gamma} \| T^{m-1} x - T^m x \| \\ &\quad + \frac{\alpha + \beta + \gamma}{1 - \alpha - 2\gamma} \| T^{n-1} x - T^n x \| \\ &\leqslant \frac{\alpha + \beta + \gamma}{1 - \alpha - 2\gamma} \{ \| x - Tx \| + \| x - Tx \| \} \end{aligned}$$

i.e.,

$$\begin{aligned} \| T^m x - T^n x \| &\leqslant \frac{2\alpha + 2\beta + 2\gamma}{1 - \alpha - 2\gamma} \cdot \| x - Tx \| \\ &\leqslant \| x - Tx \|. \end{aligned}$$

Proof of the Theorem : By Zorn's lemma, there is a minimal subset M of K, such that M is closed, convex and invariant under T. If $D(M) = 0$, then we are home. So let $D(M) > 0$. Now, since K has a normal structure, there is a point $y \in M$, such that $\operatorname{Sup} \{ \| x - y \| : x \in M \} \leqslant r < D(M)$.

Thus, $\| Ty - y \| \leqslant r$, and so by Lemma A, we get

$$D(O(y)) \leqslant r.$$

Let $N = \{ x \in M : \| x - Tx \| \leqslant r \}$ and let

$$P = C_0 (T(N)).$$

Then, P is closed, convex and nonempty. We wish to show that $T: P \to P$. Let $y \in P$.

Case 1

$$y = Tp \quad \text{for some } p \in N$$

Then, by Lemma A:

$$\| Ty - y \| = \| T^2 p - Tp \| \leqslant \| p - Tp \| \leqslant r.$$

Hence

$$y \in N \text{ and } Ty \in P.$$

Case 2

$$y = \sum_{i=1}^{n} \lambda_i \, Tp_i, \; p_i \in N, \; \lambda_i \geqslant 0, \; \sum_{i=1}^{n} \lambda_i = 1.$$

Then,

$$\| Ty - y \| = \| Ty - \sum_{i=1}^{n} \lambda_i Tp_i \|$$

$$\leqslant \sum_{i=1}^{n} \lambda_i \| Ty - Tp_i \|$$

$$\leqslant \sum_{i=1}^{n} \lambda_i \left[\frac{\alpha + \beta + \gamma}{1 - \alpha - 2\gamma} \{ \| y - Ty \| + \| p_i - Tp_i \| \} \right]$$

i.e.,

$$\| Ty - y \| \leqslant r.$$

So, $y \in N$ and $Ty \in P$

Case 3

y is the limit of terms of the form $\sum_{i=1}^{n} \lambda_i \; Tp_i$,

$$p_i \in N, \lambda_i \geqslant 0, \; \sum_{i=1}^{n} \lambda_i = 1.$$

Then,

$$\| Ty - y \| \leqslant \| Ty - \sum_{i=1}^{n} \lambda_i Tp_i \| + \| y - \sum_{i=1}^{n} \lambda_i Tp_i \|$$

Hence, from Case 2, we have

$$\| Ty - y \| \leqslant r + \| y - \sum_{i=1}^{n} \lambda_i Tp_i \|.$$

Thus, $\| Ty - y \| \leqslant r$, so $y \in N$ and $Ty \in P$. But M is minimal and so we should have $P = M$ and $D(P) = D(M)$.

Now, it can be easily shown that if A be a subset of a Banach space, then

$$D(A) = D(\bar{C}_0(A)).$$

Hence,

$$\begin{aligned} D(M) &= D(P) \\ &= D(\bar{C}_0 T(N)) \\ &= D(T(N)) \\ &= \text{Sup } \{\| Tx - Ty \| : x, y \in N\} \\ &\leqslant \text{Sup } \left\{ \frac{\alpha + \beta + \gamma}{1 - \alpha - 2\gamma} [\| x - Tx \| + \| y - Ty \|]; x, y \in N \right\} \\ &\leqslant r < D(M). \end{aligned}$$

This contradiction shows that $D(M) = 0$ and so T has a fixed point, say ξ. Obviously, ξ is unique.

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