FLOW PAST A POROUS CIRCULAR CYLINDER AT SMALL REYNOLDS NUMBER

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The matched asymptotic expansion procedure has been applied to analyse the flow past a porous circular cylinder. The flow field has been divided into the region of porous material (inside the cylinder) and the free fluid region (outside the cylinder). The inner and outer expansions have been developed in the free fluid region. The constants of integration are evaluated by the slip condition as suggested by Jones (1973) and with the conditions supplied by the matching procedure. The drag force on the cylinder has been calculated and found to decrease with increase in permeability. The drag force data obtained have been compared with those reported by Singh and Gupta (1971).

1. INTRODUCTION

Recently, Jones (1973) has modified the slip condition, as defined by Beavers and Joseph (1967) for the plane boundaries to curved surfaces. The aim of the present investigation is to see the effect of permeability and that of slip of the fluid particles at the outer surface of the porous material on the flow past a porous circular cylinder. The method of matched asymptotic expansion, as discussed by Rosenhead (1963) for an impervious circular cylinder has been utilized in the present investigation. The whole flow field has been divided into two regions :

(I) The region of porous material (inside the circular cylinder) where the flow is governed by the Darcy's law :

$$Q = -\frac{K}{a^2} \nabla P \qquad ...(1)$$

$$\nabla \cdot Q = 0 \qquad ...(2)$$

where $Q\,(= \bar{Q}/U)$ is the nondimensional filter velocity, U the uniform stream velocity, K permeability of the medium, and $P\,(= aP/\mu U)$ the non-dimensional mean pressure.

(II) The free fluid region (outside the cylinder), where the flow is governed by the Navier-Stokes' equations

$$R(q \cdot \nabla)\, q + \nabla p = \nabla^2 q. \qquad ...(3)$$

and

$$\nabla \cdot q = 0. \qquad ...(4)$$

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where

$$R = \frac{Ua}{\nu}.$$

The boundary conditions used are :

(i) The solution is bounded at $r = 0$, ...(5)

(ii) The normal velocity and pressure are continuous at $r = 1$, ...(6)

(iii) $e_{r\theta} = a\beta\,(q_\theta - Q_\theta)$ at $r = 1$, ...(7)

where

$$\beta = \frac{\alpha}{\sqrt{K}},$$

$e_{r\theta}$ is the non-dimensional rate of strain component and

q_θ, Q_θ are the tangential components of velocity.

2. Method of Solution

Now, we introduce the stream functions $\bar{\psi}$ and ψ in the porous region and in the inner region respectively, so that eqns. (1) to (4) in cylindrical polar coordinates can be written as

$$\nabla_r^2 \bar{\psi} = 0 \qquad \text{...(8)}$$

$$\nabla_r^4 \psi = -\frac{R}{r}\frac{\partial(\psi, \nabla_r^2 \psi)}{\partial(r, \theta)} \qquad \text{...(9)}$$

where

$$\nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \qquad \text{...(10)}$$

We assume the solutions of eqns. (8) and (9) in the form

$$(\bar{\psi}, \psi) = \varepsilon(R)\,[\bar{\psi}_0(r, \theta), \psi_0(r, \theta)] + \{\varepsilon(R)\}^2\,[\bar{\psi}_1(r, \theta), \psi_1(r, \theta)]$$
$$+ \{\varepsilon(R)\}^3\,[\bar{\psi}_2(r, \theta), \psi_2(r, \theta)] + \dots \qquad \text{...(11)}$$

where $\varepsilon(R)$ is an unknown function of R.

The expansions for $\bar{\psi}$ and ψ satisfy the boundary conditions (5)–(7) and the condition at infinity is replaced by the condition that the expansion for ψ should match an expansion, which is valid in the outer region. To find an outer solution, we introduce the outer variable (Oseen variables)

$$\rho = Rr \text{ and } \Psi = R\psi \qquad \text{...(12)}$$

in terms of which eqn. (9) becomes

$$\nabla_\rho^4 \Psi = -\frac{1}{\rho} \frac{\partial(\Psi, \nabla_\rho^2 \Psi)}{\partial(\rho, \theta)} \qquad ...(13)$$

where ∇_ρ^2 is the same as defined in (10) with the change that r is replaced by ρ.

We now assume the solution of (13) in the form

$$\Psi = \Psi_0(\rho, \theta) + \varepsilon(R)\Psi_1(\rho, \theta) + \{\varepsilon(R)\}^2 \Psi_2(\rho, \theta) + \dots \qquad ...(14)$$

The expansion (14) is to satisfy eqn. (13), the uniform condition at infinity, and a matching requirement for small values of ρ that it should match with the inner solution (11).

2.1. *The Leading Terms of the Expansions*

As in the earlier paper (Verma and Bhatt 1975), we may expect the leading term Ψ_0 of the expansion (4) to be the uniform stream

$$\Psi_0 = \rho \sin \theta. \qquad ...(15)$$

Similarly, $\tilde{\psi}_0$ and ψ_0 will satisfy the following equations:

$$\nabla_r^2 \tilde{\psi}_0 = 0 \qquad ...(16)$$

and

$$\nabla_r^4 \psi_0 = 0. \qquad ...(17)$$

The solutions for $\tilde{\psi}_0$ and ψ_0 may be obtained as

$$\tilde{\psi}_0 = Ar \sin \theta \qquad ...(18)$$

$$P_0 = -\frac{a^2}{K} Ar \cos \theta \qquad ...(19)$$

$$\psi_0 = (A_1 r \log r + B_1 r^3 + C_1 r^{-1} + D_1 r) \sin \theta + \sum_{n=2}^{\infty} (A_n r^{n+2} + B_n r^{2-n} + C_n r^{-n} + D_n r^n) \sin n\theta. \qquad ...(20)$$

When ψ_0 is expressed in terms of Oseen variable, the contribution to Ψ becomes

$$\varepsilon [\{A_1 \rho (\log \rho - \log R) + B_1 R^{-2} \rho^3 + C_1 R^2 \rho^{-1} + D_1 \rho\} \sin \theta + \sum_{n=2}^{\infty} \{A_n R^{-n-1} \rho^{n+2} + B_n R^{n-1} \rho^{2-n} + C_n R^{1+n} \rho^{-n} + D_n R^{1-n} \rho^n\} \sin n\theta] \qquad ...(21)$$

and the requirement that this contribution should not contain terms of greater order than unity, we have

$$B_1 = 0 \qquad ...(22)$$

$$\left.\begin{aligned} A_n = 0 \\ D_n = 0 \end{aligned}\right\} \text{ for } n \geq 2 \qquad \text{...(23)}$$

and

$$-\varepsilon \log R \to 1. \qquad \text{...(24)}$$

Further, the term of unity, i.e., $- A_1 \rho \sin \theta$ must represent the uniform stream; hence

$$A_1 = -1. \qquad \text{...(25)}$$

Thus, we get

$$\psi_0 = (C_1 r^{-1} + D_1 r - r \log r) \sin \theta + \sum_{n=2}^{\infty} (B_n r^{2-n} + C_n r^{-n}) \sin n\theta \qquad \text{...(26)}$$

and

$$p_0 = 2r^{-1} \cos \theta + \text{terms containing } B_n \text{ and } C_n. \qquad \text{...(27)}$$

Now using boundary conditions (6) and (7), the constants in (18), (19), (26) and (27) may be evaluated as

$$A = -\frac{2K}{a^2} \qquad \text{...(28)}$$

$$C_1 = -\frac{a\beta}{2(2 + a\beta)} \qquad \text{...(29)}$$

$$D_1 = \frac{a\beta}{2(2 + a\beta)} - \frac{2K}{a^2} \qquad \text{...(30)}$$

and

$$B_n = C_n = 0 \text{ for } n \geqslant 2. \qquad \text{...(31)}$$

The solution for ψ_0 can also be expressed as

$$\psi_0 = (r \log r - C_1 r^{-1} - D_1 r) \sin \theta \qquad \text{...(32)}$$

provided $\varepsilon \log R \to 1$.

2.2. *The Second Term of the Outer Expansion*

The governing equation is

$$\left(\nabla_\rho^2 - \frac{\partial}{\partial \xi}\right) \nabla_\rho^2 \Psi_1 = 0 \qquad \text{...(33)}$$

where

$$\xi = \rho \cos \theta. \qquad \text{...(34)}$$

The solution of (33), which is bounded at infinity, is

$$\nabla_\rho^2 \Psi_1 = e^{\frac{1}{2}\xi} \sum_{n=1}^{\infty} X_n K_n (\tfrac{1}{2}\rho) \sin n\theta, \qquad \text{...(35)}$$

where X_n are constants and $K_n (\frac{1}{2}\rho)$ is a modified Bessel function.

Following Proudman and Pearson (1957), the solution for ψ_1 can be obtained as

$$\Psi_1 = -\sum_{n=1}^{\infty} \phi_n\left(\frac{1}{2}\rho\right)\frac{\rho \sin n\theta}{n} + \text{harmonic function} \qquad ...(36)$$

where

$$\phi_n = 2K_1 I_n + K_0 (I_{n+1} + I_{n-1}) \qquad ...(37)$$

K_n and I_n being modified Bessel functions. Now, Ψ_1 must tend to zero for large values of ρ, and so the harmonic function in (36) can only be of the form $Y_n\rho^{-n} \sin n\theta$. The matching condition of $R\psi$ and Ψ then requires that Y_n must be zero.

2.3. *Higher Terms in the Inner and Outer Expansions*

It is easy to see that the further terms in the inner expansion will be given by

$$\left.\begin{aligned} \nabla_r^2 \tilde{\psi}_n &= 0 \\ \nabla_r^4 \psi_n &= 0 \end{aligned}\right\} \text{ for all } n. \qquad ...(38)$$

This means that in the inner expansion, the inertial terms never contribute and their effects come only through the outer boundary condition. All arguments applied for $\tilde{\psi}_0$ and ψ_0 will be applied to $\tilde{\psi}_n$ and ψ_n and it can be inferred that the latter terms will differ from the former terms only by a numerical factor. Thus, we have

$$\tilde{\psi} = \sum_{n=1}^{\infty} \varepsilon^n a_n \tilde{\psi}_0 \qquad ...(39)$$

and

$$\psi = \sum_{n=1}^{\infty} \varepsilon^n a_n \psi_0 \qquad ...(40)$$

where a_1 has already been obtained as 1.

Now, the value of ε may be calculated as suggested by Rosenhead (1963) as

$$\varepsilon = \left[\frac{1}{2} - \gamma + \log\frac{4}{R} + \frac{2K}{a^2} - \frac{\sqrt{K}}{64\,a\alpha}R^2\right]^{-1} \qquad ...(41)$$

where γ is an Euler's constant $= 0\cdot5772$.

Thus, we get

$$\tilde{\psi} = \varepsilon\left(1 + \sum_{n=1}^{\infty} a_n\varepsilon^n\right) Ar\sin\theta \qquad ...(42)$$

$$\psi = \varepsilon\left(1 + \sum_{n=1}^{\infty} a_n\varepsilon^n\right)\left[r\log r - C_1 r - D_1 r^{-1}\right]\sin\theta \qquad ...(43)$$

and for small ρ

$$\Psi = \rho \sin \theta + \varepsilon\left(\log \frac{\rho}{4} + \gamma - 1\right) \rho \sin \theta + \ldots \qquad \ldots(44)$$

We find that (43) and (44) agree in terms ρ and $\rho \log \rho$ to $0(\varepsilon)$, if $a_1 = 0$. Further, the costant a_2 has been obtained by Kaplun (1957) as $-0{\cdot}87$.

3. The Drag Force

The drag force on the porous circular cylinder is given by

$$D_r = 4\pi \mu U \varepsilon [1 - 0{\cdot}87\, \varepsilon^2 + 0(\varepsilon^3)] \qquad \ldots(45)$$

where ε is given by (41).

For $K \to 0$, eqn. (41) reduces to

$$\varepsilon = \left(\frac{1}{2} - \gamma + \log \frac{4}{R}\right)^{-1} \qquad \ldots(46)$$

and thus (45) reduces to the result obtained by Rosenhead (1963) and Kaplun (1957). Now (45) can also be written as

$$D = \frac{D_r}{4\pi\mu U} = \varepsilon [1 - 0{\cdot}87\, \varepsilon^2 + 0(\varepsilon^3)]. \qquad \ldots(47)$$

The drag is plotted against K/a^2 for different values of permeability in Fig. 1. The data have also been compared with those reported by Singh and Gupta (1971).

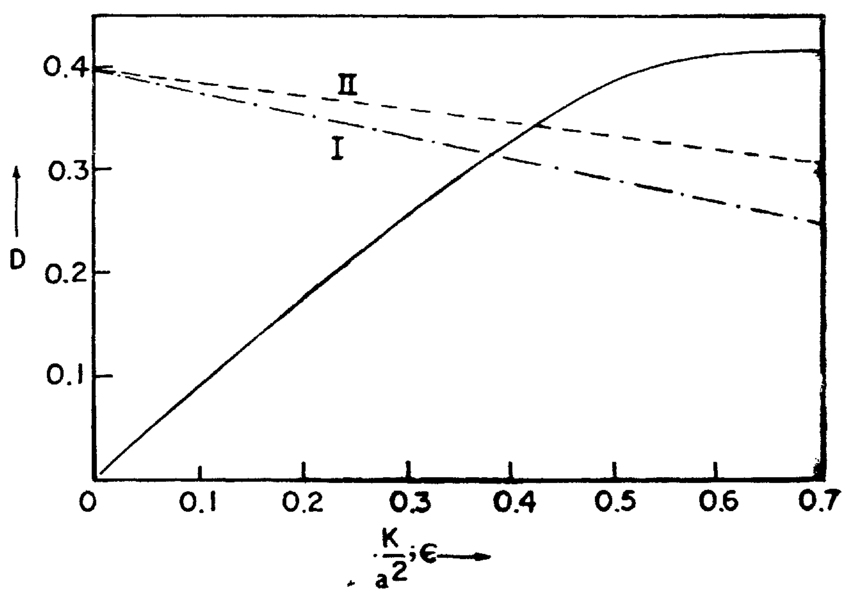

Fig. 1. (a) I—Present Paper— · — · — · — for R = 0·5

II—Singh and Gupta– – – – – – – – for R = 0·5

(b) Impermeable cylinder———— for 0 < R < 1

As expected, the drag decreases with increase in both permeability and slip. The corresponding curve in the case of solid cylinder for the Reynolds number $0 < R < 1$, is also shown in Fig. 1. It is observed that the solution breaks as $R \to 1$.

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