SOME RESULTS ON REGULAR FUNCTIONS

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Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be regular in $|z| < 1$. We shall say that $f(z) \in S_n$ if, given any complex number z_1 with $|z_1| < 1$, and any function $w(\zeta)$ univalent and satisfying $|w(\zeta)| < 1$, $w(\zeta) \neq z_1$ in $|\zeta| < 1$, we have for $\phi(\zeta) = f(w(\zeta))$, $|\phi'(0)| \leqslant 4/n(|\phi(0)| + |f(z_1)|)$.

In this study, we have obtained some results on S_n which generalize the result of Hayman (1958) (*Multivalent functions*, Cambridge University Press, London).

1. INTRODUCTION

Let S denote the class of regular and univalent functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $D = \{z : |z| < 1\}$. Bieberbach (1916) conjectured that for every function $f(z) \in S$,

$$|a_n| \leqslant n, \; n = 2, 3, 4, \ldots \qquad \ldots(1.1)$$

and the equality holds good only for functions of the form $f_\theta(z) = z(1 - e^{i\theta} z)^{-2}$. He established this result for a_2. Another result due to him is that $f(z)$ assumes every value w if $|w| < \frac{1}{4}$. These results are also known for a more general class (Hayman 1958, p. 6).

Definition 1.1—Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be regular in D. We shall say that $f(z) \in S_1$ if, given any complex number z_1 with $|z_1| < 1$, and any function $w(\zeta)$ univalent and satisfying $|w(\zeta)| < 1$, $w(\zeta) \neq z_1$ in $|\zeta| < 1$, we have for $\phi(\zeta) = f[w(\zeta)]$,

$$|\phi'(0)| \leqslant 4(|\phi(0)| + |f(z_1)|). \qquad \ldots(1.2)$$

We note that S is a subclass of S_1 (Hayman 1958). The results are as follow (Hayman 1958, p. 6).

Theorem A—Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_1$. Then

$$|a_2| \leqslant 2, \qquad \ldots(1.3)$$

Further, we have for $|z| = r$, $(0 < r < 1)$

$$\frac{r}{(1+r)^2} \leqslant |f(z)| \leqslant \frac{r}{(1-r)^2} \qquad \ldots(1.4)$$

$$|f'(z)| \leqslant \frac{1+r}{r(1-r)} |f(z)| \leqslant \frac{1+r}{(1-r)^3} . \qquad ...(1.5)$$

Finally, the equation $f(z) = w$ has exactly one root in $|z| < 1$ if $|w| < \frac{1}{4}$.

Theorem B—Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_1$ and set

$$M(r, f) = \max_{|z|=r} \quad |f(z)| \quad (0 < r < 1)$$

Then unless $f(z) = f_\theta(z) = z(1 - e^{i\theta} z)^{-2}$, $(1-r)^2 r^{-1} M(r,f)$ decreases strictly with increasing $r\,(0 < r < 1)$ and so tends to $\alpha < 1$ as $r \to 1$. Hence, the upper bounds of $|f(z)|$ and $|f'(z)|$ given by (1.4) and (1.5) respectively are attained only by the functions $f_\theta(z)$.

If in Definition 1.1, the condition (1.2) is replaced by

$$|\phi'(0)| \leqslant \frac{4}{n} (|\phi(0)| + |f(z_1)|) \qquad ...(1.6)$$

where n is a positive integer, a subclass of S_1 will satisfy the condition (1.6). Let us denote this class by S_n. Furthermore, if we replace the condition (1.2) in Definition 1.1 by

$$|\phi'(0)| \leqslant 2^{2/n} (|\phi(0)| + |f(z_1)|) \qquad ...(1.7)$$

we get another subclass of S_1, say S'_n. It is clear that $S_n \subset S'_n \subset S_1$.

The object of this paper is to generalize Theorems A and B for the class S_n.

§ 2. We now prove the following theorems :

Theorem 2.1—Suppose $f(z) \in S_n$. Then for $|z| = r < 1$

$$\frac{r}{(1+r^n)^{2/n}} \leqslant \frac{2^{2/n}\, n}{4} \frac{r}{(1+r^n)^{2/n}} \leqslant |f(z)| \leqslant \frac{r}{(1-r^n)^{2/n}} \qquad ...(2.1)$$

$$|f'(z)| \leqslant \frac{1+r^n}{r(1-r^n)} |f(z)| \leqslant \frac{1+r^n}{(1-r^n)^{2/n+1}} \qquad ...(2.2)$$

PROOF : Let

$$\frac{z}{(1-z^n)^{2/n}} = Z = \frac{2^{2/n}\, d\zeta}{(1-\zeta^n)^{2/n}} \qquad ...(2.3)$$

where $d = r/(1 + r^n)^{2/n}$ for some fixed r satisfying $0 < r < 1$. Let $|z| < 1$ be cut along the line segments $re^{\{(2k+1)/n\}\pi i}$ to $e^{\{(2k+1)/n\}\pi i}$, $k = 0, 1, 2, \ldots, n-1$. Let Z-plane be cut along the rays $\arg Z = (2k+1)\pi/n$ commencing from the points $Z_k = d\; e^{\{(2k+1)/n\}\pi i}$, $k = 0, 1, 2, \ldots, n-1$. We denote this remaining plane by Z'. We can see from (2.3) that the remaining portion of $|z| < 1$, thus obtained, will be mapped (1, 1) conformally onto Z'. Also $|\zeta| < 1$ is mapped (1, 1) conformally onto Z'. Thus, (2.3) defines an univalent function

$z = w(\zeta)$, which has the properties $|w(\zeta)| < 1$ and $w(\zeta) \neq re^{\{(2k+1)/n\}\pi i}$ for $|\zeta| < 1$. Since, $f(z) \in S_n$, we have

$$|\phi'(0)| \leqslant \frac{4}{n}(|\phi(0)| + |f(re^{\{(2k+1)/n\}\pi i})|) \qquad \ldots(2.4)$$

where

$$\phi(\zeta) = f(w(\zeta)).$$

From (2.3), we see that $\zeta = 0$, $z = 0$. Hence, $\phi(0) = 0$. Also, it enables us to write

$$|\phi'(0)| = 2^{2/n} d \leqslant \frac{4}{n} |f(re^{\{(2k+1)/n\}\pi i})| \qquad \ldots(2.5)$$

i.e.,

$$|f(re^{\{(2k+1)/n\}\pi i})| \geqslant \frac{2^{2/n} n}{4} \frac{r}{(1 + r^n)^{2/n}}$$

Since $e^{-i\theta} f(ze^{i\theta}) \in S_n$ if $f(z) \in S_n$, we get

$$|f(re^{i\theta})| \geqslant \frac{2^{2/n} n}{4} \frac{r}{(1 + r^n)^{2/n}} \geqslant \frac{r}{(1 + r^n)^{2/n}}$$

and this is the left hand side of (2.1).

Now, let

$$Z = \frac{z}{(1 - z^n)^{2/n}} = K\left(\frac{1 + \zeta}{1 - \zeta}\right)^{2/n} \qquad \ldots(2.6)$$

where $K = r/(1 - r^n)^{2/n}$. Here, r is a fixed number satisfying the inequality $0 < r < 1$. Let us consider the open sector Ω_n in $|z| < 1$ bounded by the lines arg $z = \pm\, \pi/n$, which subtend an angle $2\pi/n$ at the origin. (2.6) will map Ω_n (1, 1) conformally onto Z-plane bounded by the lines arg $Z = \pm\, \pi n$, which subtend an angle $2\pi/n$ at the origin. Also, (2.6) ensures that $|\zeta| < 1$ is mapped (1, 1) conformally onto the region in Z-plane bounded by the lines arg $Z = \pm\, \pi/n$ and subtending an angle $2\pi/n$ at the origin (Nehari 1952, p. 237). In the above case, we consider the principal branch for which $1^{1/n} = 1$. Thus, (2.6) defines an univalent function $z = w(\zeta)$ such that $|w(\zeta)| < 1$ and $w(\zeta) \neq 0$ for $|\zeta| < 1$. Since $f(z) \in S_n$, the condition (1.6) will stand true and hence making use of (2.6), we get

$$|f'(r)| \cdot \frac{4}{n} \cdot \frac{r(1 - r^n)}{1 + r^n} \leqslant \frac{4}{n} |f(r)|. \qquad \ldots(2.7)$$

Since $e^{-i\theta} f(ze^{i\theta}) \in S_n$, we conclude that

$$\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| \leqslant \frac{1 + r^n}{r(1 - r^n)} \qquad (2.8)$$

and this is the left-hand inequality of (2.2)

Now,

$$\frac{\partial}{\partial r} \log |f(re^{i\theta})| \leqslant \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| \leqslant \frac{1+r^n}{r(1-r^n)}. \quad ...(2.9)$$

Integrating (2.9) for r_1 to r_2, where $0 < r_1 < r_2 < 1$, we deduce that

$$\log \left| \frac{f(r_2 e^{i\theta})}{f(r_1 e^{i\theta})} \right| \leqslant \int_{r_1}^{r_2} \frac{1+r^n}{r(1-r^n)}\, dr = \log \frac{r_2 (1-r_1^n)^{2/n}}{r_1 (1-r_2^n)^{2/n}}$$

or

$$\frac{(1-r_2^n)^{2/n}}{r_2} |f(r_2 e^{i\theta})| \leqslant \frac{(1-r_1^n)^{2/n}}{r_1} |f(r_1 e^{i\theta})|. \quad ...(2.10)$$

Letting $r_1 \to 0$ and replacing r_2 by r, we get the right-hand inequality of (2.1). Also, (2.2) follows from (2.1) and (2.9).

As in the proof of the above theorem, we have the following result.

Theorem 2.2—Let $f(z) \in S'_n$. Then for $|z| = r < 1$,

$$|f(z)| \geqslant r(1+r^n)^{-2/n}. \quad ...(2.11)$$

Theorem 2.3—Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S'_n$. Then

$$a_i = 0 \text{ for } i = 2, 3, \ldots, n \quad ...(2.12)$$

and

$$|a_{n+1}| \leqslant 2/n \quad ...(2.13)$$

Also, the equation $f(z) = w$ has unique root in $|z| < 1$ if $|w| < \frac{1}{2^{2/n}}$.

Proof : Let us choose θ, so that $a_2 e^{i\theta} = -|a_2|$. Then, as $r \to 0$

$$\begin{aligned} |f(re^{i\theta})| &= |r + a_2 e^{i\theta} r^2 + 0(r^3)| \\ &= r - |a_2| r^2 + 0(r^3). \end{aligned} \quad ...(2.14)$$

But from (2.11), we have

$$|f(re^{i\theta})| \geqslant r - \frac{2}{n} r^{n+1} + 0(r^{2n+1}). \quad ...(2.15)$$

(2.14) and (2.15) imply that $a_2 = 0$. Similarly, we can show that $a_3 = a_4 = \ldots = a_n = 0$ and

$$|a_{n+1}| \leqslant 2/n. \quad (2.16)$$

Again, from the inequality (2.11) we see that $f(z) \neq 0$ in $|z| < 1$, except $z = 0$. Let $|w| < r/(1+r^n)^{2/n}$, then due to Rouche's theorem (Nehari 1962, p. 131) we can say that $f(z) = w$ will have unique root in the circle $|z| < r$. If $r \to 1$, we get the required result, i.e., $f(z) = w$ has unique root in $|z| < 1$ if $|w| < 1/2^n$.

Note : The above result is also true for the class S_n, because $S_n \subset S'_n$.

Theorem 2.4—Suppose $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_n$ and set

$$M(r,f) = \max_{|z|=r} |f(z)| \, (0 < r < 1).$$

Then unless $f(z) = f_\theta(z) = z(1 - e^{i\theta} z^n)^{-2/n}$, $(1 - r^n)^{2/n} r^{-1} M(r, f)$ decreases strictly with increasing r $(0 < r < 1)$ and so tends to $\alpha < 1$ as $r \to 1$. Hence, the upper bounds for $|f(z)|$ and $|f'(z)|$ given by (2.1) and (2.2) respectively are attained only by the function $f_\theta(z)$.

PROOF : To prove this, we can see that equality in (2.10) can hold only if the equality holds in both the inequalities of (2.9) for $r_1 < r < r_2$. This gives

$$\operatorname{Re}\left\{\frac{e^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})}\right\} = \frac{1 + r^n}{r(1 - r^n)} \qquad \text{...(2.17)}$$

and so

$$\operatorname{Im}\left\{\frac{e^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})}\right\} = 0 \qquad \text{...(2.18)}$$

i.e.,

$$\frac{zf'(z)}{f(z)} = \frac{1 + z^n e^{-in\theta}}{1 - z^n e^{-in\theta}}$$

for $z = re^{i\theta}$ $(r_1 < r < r_2)$, and so by analytic continuation it holds throughout $|z| < 1$. In this case, $f(z) = f_{-n\theta}(z)$. Otherwise, strict inequality holds good in (2.10) for $0 < r_1 < r_2 < 1$, $0 \leqslant \theta \leqslant 2\pi$. Choose θ so that $|f(r_2 e^{i\theta})| = M(r_2, f)$. (2.10) gives

$$\frac{(1 - r_2^n)^{2/n}}{r_2} M(r_2, f) < \frac{(1 - r_1^n)^{2/n}}{r_1} |f(r_1 e^{i\theta})|$$

$$\leqslant \frac{(1 - r_1^n)^{2/n}}{r_1} M(r_1, f).$$

Hence, unless $f(z) = f_\theta(z)$, $\psi(r) = (1 - r^n)^{2/n} r^{-1} M(r, f)$ decreases strictly with increasing r $(0 < r < 1)$. But $\psi(r) \leqslant 1$ follows from (2.1). Thus, $\psi(r) < 1$ $(0 < r < 1)$ and so tends to the limit $\alpha < 1$. It is also established that the upper bounds for $|f(z)|$ and $|f'(z)|$ are attained only for functions $f_\theta(z)$.

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