# A PROCEDURE FOR TIME MINIMIZATION TRANSPORTATION PROBLEM

*by* H. L. BHATIA, *Department of Mathematics, P.G.D.A.V. College, New Delhi* 110 024

KANTI SWAROOP, *Indian Institute of Public Administration, New Delhi*

*and*

M. C. PURI, *Department of Mathematics, Hans Raj College, Delhi University, Delhi* 110 007

(*Received* 8 *September* 1975; *after revision* 14 *July* 1976)

This paper develops a technique for minimizing time in a transportation problem. The procedure involves finite iterations and is based on moving from a basic feasible solution to another till the last solution is arrived at. A numerical example illustrating the method is also included.

## INTRODUCTION

In a time minimizing transportation problem, the time of transporting goods from $m$ origins to $n$ destinations is minimized, satisfying certain conditions in respect of availabilities at sources and requirements at the destinations.

Thus, a time minimizing transportation problem is:

Minimize

$$Z = [\underset{(i,j)}{\text{Max}} \quad t_{ij}/x_{ij} > 0]$$

subject to

$$\left.\begin{aligned} \sum_{j=1}^{n} x_{ij} &= a_i, \; i = 1, 2, \ldots m \\ \sum_{i=1}^{m} x_{ij} &= b_j, \; j = 1, 2, \ldots n \end{aligned}\right] \quad (1)$$

$$x_{ij} \geqslant 0$$

Problem (I)

Here $t_{ij}$ is the time of transporting goods from the $i$th origin, where availability is $a_i$ to the $j$th destination, where the requirement is $b_j$. For any given feasible solution, $X = [x_{ij}]$ satisfying (1), the time of transportation is the maximum of $t_{ij}$'s among the cells in which there are positive allocations, i.e., corresponding to the solution $X$, the time of transportation is

$$[\underset{(i,j)}{\text{Max}} \quad t_{ij}/x_{ij} > 0].$$

The aim is to minimize this time of transportation. Such problems arise when it is required to transport perishable goods or, during war days, it is required to transport food and armaments in the shortest possible time and in so many other similar situations.

Thus, the basic difference between the cost minimizing transportation problem and the present problem is that whereas the cost of transportation changes with variations in the quantity of the commodity, the time involved remains unchanged, irrespective of the quantities of the commodity involved in the occupied cells in the present problem.

It is assumed in the given problem that (i) the carriers have sufficient capacity to carry goods from an origin to a destination in a single trip, and (ii) they start simultaneously from their respective origins.

The time minimizing transportation problem has been studied by Hammer (1969), Garfinkel and Rao (1971) and Szwarc (1971). Hammer (1969) and Szwarc (1971) used lebelling techniques to solve the problem (I). Garfinkel and Rao (1971) solved the problem by introducing a sufficiently large cost $M$ on certain routes. In the present study, the labelling technique has not been used and the costs associated are just 0 and 1 and do not involve costs like $M$, as was done by Garfinkel and Rao (1971). Thus, the solution method presented in this paper is quite simple from computational point of view.

## THEORETICAL DEVELOPMENT

*Definitions*

(i) *A feasible solution*—A set $X = [x_{ij}]$ of non-negative variables satisfying (1) is called a feasible solution.

(ii) *A better feasible solution*—Let $X^1 = [x^1_{ij}]$ and $X^2 = [x^2_{ij}]$ be two feasible solutions of problem (I).

Let

$$M^1 = [(i,j)/x^1_{ij} > 0],\ M^2 = [(i,j)/x^2_{ij} > 0]$$

$$T^1 = \max_{(i,j) \in M^1} t_{ij} \qquad T^2 = \max_{(i,j) \in M^2} t_{ij}$$

$$Q^1 = [(i,j)/t_{ij} = T^1,\ (i,j) \in M^1]$$

$$Q^2 = [(i,j)/t_{ij} = T^2,\ (i,j) \in M^2]$$

A solution $X^2$ is said to be better than $X^1$, if

either (1) $T^2 < T^1$,

or

(2) $T^2 = T^1, \quad \sum_{(i,j) \in Q^2} x^2_{ij} < \sum_{(i,j) \in Q^1} x^1_{ij}$

in case (1) time is improved,

in case (2) the amount of commodity transported at time $T^2$ in solution $X^2$ is less than that at time $T^1$ in solution $X^1$ (Hammer 1969).

*Almost Optimal Solution*

A feasible solution $X = [x_{ij}]$ for which $[\underset{(i,j)}{\text{Max}}\ t_{ij}\ / x_{ij} > 0]$ is minimal is called an almost optimal solution.

*Optimal Solution*

A feasible solution $X = [x_{ij}]$ of the Problem (I) is said to be the optimal solution if there does not exist any better solution, i.e., for which the time of transportation $T$ is the least, and $\sum_{(i,j) \in Q} x_{ij}$ also has the minimum value.

*Theorem* 1—If there is a feasible solution to a set of simultaneous equations $AX = b$, $X \geqslant 0$, then there is a basic feasible solution (refer Hadley 1962, Arts. 3.4, p. 80).

Hadley (1962) proved this theorem by reducing the number of positive variables in the given feasible solution one by one till the columns of $A$ associated with positive variables are linearly independent.

In the proof of this theorem, the set of positive variables in the basic feasible solution is a subset of the positive variables in the given feasible solution. The values of the variables in the two sets, however, may be different.

*Theorem* 2—There exists a basic feasible solution which is optimal for Problem (I).

PROOF : Let $[t_{\alpha\beta}]$ represent a set of $t_{ij}$'s corresponding to those $x_{ij}$'s that form a feasible solution. Let $[t_{\gamma\delta}]$ be a set of $t_{ij}$'s corresponding to those positive $x_{ij}$'s which form the basic feasible solution derived from the feasible solution. Since $[t_{\gamma\delta}]$ is essentially a subset of $[t_{\alpha\beta}]$, the maximum of $t_{ij}$'s in $[t_{\gamma\delta}]$ is less than or equal to the maximum of $t_{ij}$'s in $[t_{\alpha\beta}]$. It follows that if there exists a feasible solution of (I) with the time of transportation $T$, there also exists a basic feasible solution with the time of transportation $T'$, such that $T' \leqslant T$. Thus, if there exists an optimal feasible solution with the time of transportation $T$, there also exists an optimal basic feasible solution with the same time $T$ of transportation. Hence, there always exists a basic feasible solution optimal for Problem (I).

A basic feasible solution is said to be locally optimal if there does not exist any better adjacent basic feasible solution.

*Theorem* 3—A locally optimal solution to Problem (I) is also globally optimal.

PROOF : Consider a locally optimal solution $X^1 = [x^1_{ij}]$ of Problem (I). If possible, let there be a better solution, say $X^2 = [x^2_{ij}]$. For these solutions define $M^1$, $M^2$, $T^1$, $T^2$, $Q^1$, $Q^2$ as done earlier.

Let

$$p^1 = \sum_{(i,j) \in Q^1} x^1_{ij},\quad p^2 = \sum_{(i,j) \in Q^2} x^2_{ij}$$

Consider a profit-maximization problem defined by constraints (1) and the profit matrix $[c_{ij}]$, where

$$c_{ij} = \begin{bmatrix} 1 & \text{if } t_{ij} < T^1 \\ 0 & \text{if } t_{ij} \geqslant T^1 \end{bmatrix}$$

(the cells in which the time of transportation is greater than or equal to $T^1$ are considered non-economical carrying no profit).

$$C^1 = \text{Profit yielded by solution } X^1 = \sum_{(i,\, j)\, \in M^1} \sum c_{ij}\, x_{ij}^1$$

$$= \sum_{(i,\, j)\, \in M^1/Q^1} \sum x_{ij}^1 = N - p^1$$

where $N = \sum_i a_i = \sum_j b_j$

$$C^2 = \text{Profit yielded by solution } X^2 = \sum_{(i,\, j)\, \in M^2} \sum c_{ij}\, x_{ij}^2.$$

As $X^2$ is a better solution than $X^1$, either $T^2 < T^1$ or $T^2 = T^1$, $p^2 < p^1$. When $T^2 < T^1$, $c_{ij} = 1$ for all the occupied cells of $X^2$,

$$\therefore\ C^2 = \sum_{(i,\, j)\, \in M^2} c_{ij}\, x_{ij}^2 = \sum_{(i,\, j)\, \in M^2} x_{ij}^2 = N.$$

Since $N > N - p^1$, $C^2 > C^1$.

When $T^2 = T^1$, $p^2 < p^1$.

$$C^2 = \sum_{(i,\, j)\, \in M^2} \sum c_{ij}\, x_{ij}^2 = \sum_{(i,\, j)\, \in M^2/Q^2} \sum x_{ij}^2 = N - p^2.$$

As $N - p^2 > N - p^1$, $C^2 > C^1$.

It follows that $X^1$ is not the optimal solution for the profit-maximization transportation problem. Therefore, there exists an adjacent basic feasible solution $X^3$ for the profit maximization problem better than $X^1$, such that $C^3 > C^1$, where $C^3$ is the profit yielded by solution $X^3$. Define $M^3$, $T^3$, $Q^3$, $p^3$ in the usual manner for the solution $X^3$.

It can be shown that $T^3 \leqslant T^1$.

If possible, let $T^3 > T^1$. Now there may exist a possibility in which $C^3$ contains all the cells of $X^1$ with time $T^1$ and the allocation same as in $X^1$.

Then

$$C^3 = \sum_{(i,\, j)\, \in M^3} \sum c_{ij}\, x_{ij}^3 = \sum_{(i,\, j)\, \in M^3/Q^3 \cup Q^1} \sum x_{ij}^3 = N - p^3 - p^1$$

As $C^1 = N - p^1$ and $N - p^1 > N - p^3 - p^1$

$$\Rightarrow C^1 > C^3 \text{ or } C^3 < C^1$$

contradicting the fact that $X^3$ is an adjacent basic feasible solution better than $X^1$ for the profit-maximization problem.

Therefore, $T^3 \not> T^1$. Hence, $T^3 \leqslant T^1$.

where

$$T^3 = T^1$$

$$C^3 = \sum_{(i,j) \in M^3}\sum c_{ij} x_{ij}^3 = \sum_{(i,j) \in M^3/Q^3}\sum x_{ij}^3 = N - p^3$$

$$C^3 > C^1 \Rightarrow N - p^3 > N - p^1$$

$$\Rightarrow p^3 < p^1.$$

Thus, either $T^3 < T^1$ or $T^3 = T^1, p^3 < p^1$. Thus, $X^3$ is a better adjacent basic feasible solution with respect to time, implying that $X^1$ is not locally optimal. This contradicts the hypothesis. Hence, a locally optimal solution is also globally optimal.

*Note* : This Theorem was also established by Hammer (1969) by a different construction of the matrix $[c_{ij}]$ suitable in accordance with the development of theory.

## ALGORITHM

The algorithm consists of three main steps :

(1) Determination of an initial basic feasible solution which can be found by the methods applicable in the case of the common cost minimizing transportation problem; (2) finding an adjacent better basic feasible solution, and (3) repetition of step (2) till no better adjacent basic feasible solution can be found.

Step (2) deals with the determination of a cell not in the basis which, when introduced, will either reduce the time of transportation or reduce the allocation in at least one of the cells $\varepsilon$ $Q$, where $Q$ is the set of cell with positive allocations and corresponding time equal to the time of transportation.

Thus, step (2) consists of the following substeps :

(2.1) Determination of the set $S$ of all the cells not in the basis such that if any one of these cells is introduced into the basis, it would reduce $\sum_{(i,j) \in Q} x_{ij} = p$ in amount or to zero.

(2.2) Choosing among the elements of $S$ the one, say $(i_0, j_0)$, for which $t_{i_0, j_0}$ is minimum.

Substep (2.2) determines the cell $(i_0, j_0)$ to enter the basis.

Substep (2.1) determines the set $S$ of cells eligible for entry and this set is determined as follows :

Let $X = [x_{ij}]$ be any basic feasible solution of Problem (I), yielding the time $T$ of transportation.

Let

$$M = [(i,j)/x_{ij} > 0]$$

$$Q = [(i,j)/t_{ij} = T, (i,j) \in M]$$

Define a profit matrix $[c_{ij}]$ as in Theorem 2, i.e.,

$$c_{ij} = \begin{cases} 1 \text{ if } t_{ij} < T \\ 0 \text{ if } t_{ij} \geqslant T. \end{cases}$$

For this profit-maximization problem, determine $u_i$ for rows and $v_j$ for columns in the usual manner choosing, say $u_h = 0$, where $(h, k) \in Q$ having maximum allocation (Hadley 1962). Since profit is to be maximized at time $T$, the cells eligible for entry into the basis are those for which $\Delta_{rs} > 0$, where

$$\Delta_{rs} = [c_{rs} - (u_r + v_s)/(r, s) \notin B],$$

where $B$ is the set of basic cells. A cell $(r, s)$ with $t_{rs} \geq T$, $c_{rs} = 0 \notin S$, for if such a cell enters the basis at a positive level, the profit decreases, but our problem is a profit maximization problem. In case it enters at zero level, neither time nor profit changes, so that it is an alternative degenerate solution. Thus, such cells are not eligible for entry into the basis. For cells $(r, s)$ with $t_{rs} < T$, $c_{rs} = 1$, $\Delta_{rs}$ is positive if an only is $u_r + v_s \leqslant 0$.

$$\therefore S = [(r, s)/(r, s) \notin B, \; u_r + v_s \leqslant 0, \quad C_{rs} = 1]$$

The procedure is bound to converge, for it involves movement from one basic feasible solution to another better basic feasible solution; the solutions are always finite in number. The process terminates when $S = \Phi$.

*Note*—An imbalanced transportation problem, when $\sum_i a_i \neq \sum_j b_j$, can be solved by introducing either a dummy destination (when $\sum_i a_i > \sum_j b_j$) with demand $\sum_i a_i - \sum_j b_j$ or a dummy source (when $\sum_i a_i < \sum_j b_j$) with a capacity $\sum_j b_j - \sum_i a_i$. The time of transportation to dummis will be less than the minimum $t_{ij}$ in the time matrix $[t_{ij}]$.

## Numerical Illustration

Let there be six producers $O_i$ supplying 15, 7, 45, 30, 12, and 16 units of goods respectively with seven consumers $D_j$ demanding 20, 13, 11, 27, 9, 5, and 40 units respectively and the following table gives data regarding transportation time $t_{ij}$, $i = 1, 2, \ldots, 6$, $j = 1, 2, \ldots, 7$.

| Consumer → <br> ↓ Producer | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ | $D_7$ | $a_i$ |
|---|---|---|---|---|---|---|---|---|
| $O_1$ | 12 | 13 | 34 | 7 | 8 | 29 | 19 | 15 |
| $O_2$ | 7 | 18 | 36 | 40 | 38 | 6 | 10 | 7 |
| $O_3$ | 11 | 20 | 30 | 21 | 21 | 29 | 31 | 45 |
| $O_4$ | 27 | 12 | 39 | 31 | 5 | 36 | 12 | 30 |
| $O_5$ | 15 | 17 | 32 | 36 | 22 | 16 | 14 | 12 |
| $O_6$ | 17 | 38 | 16 | 33 | 23 | 30 | 29 | 16 |
| $b_j$ | 20 | 13 | 11 | 27 | 9 | 5 | 40 | 125 |

An initial feasible solution $X^1$ is given below:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ | $D_7$ | $a_i$ | $u_i$ |
|---|---|---|---|---|---|---|---|---|---|
| $O_1$ | 12 **15** | 13 | 34 | 7 | 8 | 29 | 19 | 15 | 0 |
| $O_2$ | 7 **2** | 18 | 36 | 40 | 38 | 6 **5** | 10 | 7 | 0 |
| $O_3$ | 11 **3** | 20 **4** | 30 **11** | 21 **27** | 21 | 29 | 31 | 45 | 0 |
| $O_4$ | 27 | 12 **9** | 39 | 31 | 5 **9** | 36 | 12 **12** | 30 | 0 |
| $O_5$ | 15 | 17 | 32 | 36 | 22 | 16 | 14 **12** | 12 | 0 |
| $O_6$ | 17 | 38 | 16 | 33 | 23 | 30 | 29 **16** | 16 | 0 |
| $b_j$ | 20 | 13 | 11 | 27 | 9 | 5 | 40 | | |
| $v_j$ | 1 | 1 | 0 | 1 | 1 | 1 | 1 | | |

Bold numerals denote the allocations in the basic cells of the solution.

$$M^1 = [(i,j)/x_{ij} > 0]$$
$$= [(1,\ 1),\ (2,\ 1),\ (2,\ 6),\ (3,\ 1),\ (3,\ 2),\ (3,\ 3),\ (3,\ 4),\ (4,\ 2)$$
$$(4,\ 5),\ (4,\ 7),\ (5,\ 7),\ (6,\ 7)]$$

$T^1 = \underset{(i,j)\in M^1}{\text{Max}}\ t_{ij} = 30$ [corresponding to cell (3, 3)] Define profit matrix $[c_{ij}]$ as

$$c_{ij} = \begin{bmatrix} 1 \text{ if } t_{ij} < 30 \\ 0 \text{ if } t_{ij} \geqslant 30 \end{bmatrix}$$

$$Q^1 = [(i,j)/t_{ij} = T^1,\ (i,j) \in M] = [(3.\ 3)]$$

$$p^1 = \sum_{(i,j)\in Q^1} x^1_{ij} = 11$$

$$S = [(r,s)/u_r + v_s \leqslant 0,\ (r,s) \notin B^1,\quad c_{rs} = 1] = [f(6,\ 3)]$$

cell to enter $= [(r,s)/(r,s) \in S,\ t_{rs}$ is the minimum of $t_{ij}$'s, where $(i,j) \in S] = (6,\ 3)$.

Thus, (6, 3) enters the basis.

The new solution $X^2$ is given by :

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ | $D_7$ | $a_i$ | $u_i$ |
|---|---|---|---|---|---|---|---|---|---|
| $O_1$ | 12 **15** | 13 | 34 | 7 | 8 | 29 | 19 | 15 | 0 |
| $O_2$ | 7 **2** | 18 | 36 | 40 | 38 | 6 **5** | 10 | 7 | 0 |
| $O_3$ | 11 **3** | 20 **13** | 30 **2** | 21 **27** | 21 | 29 | 31 | 45 | 0 |
| $O_4$ | 27 | 12 | 39 | 31 | 5 **9** | 36 | 12 **21** | 30 | 1 |
| $O_5$ | 17 | 17 | 32 | 36 | 22 | 16 | 14 **12** | 12 | 1 |
| $O_6$ | 15 | 38 | 16 **9** | 33 | 23 | 30 | 29 **7** | 16 | 1 |
| $b_j$ | 20 | 13 | 11 | 27 | 9 | 5 | 40 | | |
| $v_j$ | 1 | 1 | 0 | 1 | 0 | 1 | 0 | | |

This solution gives $T^2 = 30$, $p^2 = 2$,

$\therefore$ $T^2 = T^1$, $p^2 < p^1$, i.e., allocation in the cell (3, 3) is reduced. Proceeding like this, the following solutions are obtained :

$X^3$ yielding $T^3 = 29$, $p^3 = 5$

$X^4$ yielding $T^4 = 21$, $p^4 = 27$

$X^5$ yielding $T^5 = 21$, $p^5 = 19$

$X^6$ yielding $T^6 = 21$, $p^6 = 17$

The solution $X^6$ is given below :

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ | $D_7$ | $a_i$ | $U_i$ |
|---|---|---|---|---|---|---|---|---|---|
| $O_1$ | 2 | 3 | 34 | 7 **10** | 8 **5** | 29 | 19 | 15 | 1 |
| $O_2$ | 7 | 18 | 36 | 40 | 38 | 6 **5** | 10 **2** | 7 | 1 |
| $O_3$ | 11 **15** | 20 **13** | 30 | 21 **17** | 21 | 29 | 31 | 45 | 0 |
| $O_4$ | 27 | 12 | 39 | 31 | 5 **4** | 36 | 12 **26** | 30 | 1 |
| $O_5$ | 17 | 17 | 32 | 36 | 22 | 16 | 14 **12** | 12 | 1 |
| $O_6$ | 15 **5** | 38 | 16 **11** | 33 | 23 | 30 | 29 | 16 | 0 |
| $b_j$ | 20 | 13 | 11 | 27 | 9 | 5 | 40 | | |
| $v_j$ | 1 | 1 | 1 | 0 | 0 | 0 | 0 | | |

$S = [\Phi]$. Thus, the process terminates.

Hence, the optimal time for the transportation problem is $T = 21$, and at time 21, the solution is :

$$X_{14} = 10, \ x_{15} = 5, \ x_{26} = 5, \ x_{27} = 2,$$

$$x_{31} = 15, \ x_{32} = 13, \ x_{34} = 17, \ x_{45} = 4,$$

$$x_{47} = 26, \ x_{57} = 12, \ x_{61} = 5, \ x_{63} = 11.$$

## ACKNOWLEDGEMENT

The authors are grateful to the referee for the vauable suggestions for the improvement of this paper. Their thanks are also due to Shri Sri Ram Arora of Hans Raj College, Delhi University, Delhi, who participated in the discussion during thedevelopment of this paper and to Shri J. K. Kapoor, Hans Raj College, Delhi University, Delhi, for extending all co-operation and encouragement.

## References

Garfinkel, R. S., and Rao, M. R. (1971). The bottleneck transportation problem. *Nav. Res. Logist. Q.*, **18**, 465–72.
Hadley, G. (1962). Linear Programming. Addison Wesley Publishing Co., Inc., Reading, Massachusetts.
Hammer, P. L. (1969). Time minimizing transportation problems. *Nav. Res. Logist. Q.*, **16**, 345–57.
Szwarc, W. (1971). Some remarks on the transportation problem. *Nav. Res. Logist. Q.*, **18**, 473–85.