σ-REFLEXIVE PRIMARY SEMIGROUPS

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Satyanarayana (1972) has initiated the study of commutative primary semigroups, and commutative semiprimary semigroups have been discussed by Lal (1975). We discuss in the present paper these semigroups in a very general set-up, namely when the underlying semi-group is σ-reflexive. The concept of σ-reflexive semigroups is due to Chacron and Thierrin (1972). We prove that a σ-reflexive semigroup is semiprimary if its ideals are totally ordered; idempotents form a chain in a σ-reflexive semigroup. In a regular σ-reflexive semigroup, primary and semiprimary semigroups become identical, but not in general. Some equivalent conditions in a left cancellative σ-reflexive semigroup have also been obtained. Finally, the inter-relations between the primary semigroups and the Archimedean semigroups (defined upon σ-reflexive semigroups) have been obtained.

1. PRELIMINARIES AND DEFINITIONS

Definition 1.1—A semigroup S is called σ-reflexive if for any sub-semigroup H of S, $ab \in H$ $(a, b \in S)$ implies that $ba \in H$. Clearly, a commutative semigroup is a σ-reflexive semigroup.

The following two results on σ-reflexive semigroups are due to Chacron and Thierrin (1972).

Proposition 1.2—Any semigroup S is σ-reflexive if it satisfies the condition: $\forall\, a,\ b \in S,\ \exists\, m = m(a,\ b) \geq 1,\ ab = (ba)^m$.

Proposition 1.3—Let a, b be any two non-commuting elements of a σ-reflexive semigroup S. Then, for some $m > 1$, $ab = (ab)^m$.

Definition 1.4—Let A be any ideal of a semigroup S. We call radical (Zarishi and Samuel 1958) of A (to be denoted by $\sqrt{A}$) as the set $\{x \in S : x^n \in A \text{ for some } n\}$. Clearly $A \subseteq \sqrt{A}$.

Proposition 1.5—Radical of an ideal A of a σ-reflexive semigroup S is an ideal.

PROOF : Let $x \in \sqrt{A}$ and $a \in S$. Then, $x^n \in A$ for some n. Now, $xa = (ax)^m$ for some $m \geq 1$, by Proposition 1.2. If $m = 1$, then $xa = ax$ and clearly $xa \in \sqrt{A}$. Now, suppose that $m > 1$; we have $xa = (ax)^m = a.\ (xa)^{m-1}.\ x = a.\ (ax)^{m(m-1)},\ x = a^2 .\ (xa)^{m(m-1)-1} .\ x^2$, and repeating this procedure, we ultimately get that $xa \in A$ and hence in $\sqrt{A}$. By a simil arargument, $ax \in \sqrt{A}$, proving that $\sqrt{A}$ is an ideal of S, in case S is σ-reflexive.

Remark 1.6—The σ-reflexive character of S is an essential part of the hypothesis to make $\sqrt{A}$ an ideal of S. For instance, consider, $S = \{0, a_1, a_2, a_3, a_4\}$, where multiplication is given by the table :

	0	a_1	a_2	a_3	a_4
0	0	0	0	0	0
a_1	0	a_1	0	a_3	0
a_2	0	0	a_2	0	a_4
a_3	0	0	a_3	0	a_1
a_4	0	a_4	0	a_2	0

This semigroup is not σ-reflexive, as $H = \{a_2\}$ is a subsemigroup of S containing $a_4 \,.\, a_3$, but not $a_3\, a_4$. Now taking $A = \{0\}$, we have $\sqrt{A} = \{0, a_3, a_4\}$, but this is not an ideal of S (in fact, this is not even a subsemigroup of S).

Definition 1.7—An ideal Q of a semigroup S is said to be primary (Zariski and Samuel 1958) iff $ab \in Q$, $a \notin Q\,(a, b \in S) \Rightarrow b^n \in Q$ for some n.

Theorem 1.8—Let S be a σ-reflexive semigroup. Then, an ideal P of S is prime if it is completely prime.

PROOF : Let P be prime and $ab \in P\,(a, b \in S)$. For any x in S, $xb = (bx)^m$ for some $m \geqslant 1$. But then $axb = a\,(bx)^m \in P$ for all $x \in S$. Hence, $aSb \subseteq P$, whence $a \in P$ or $b \in P$, since P is prime. Therefore, P is completely prime. The converse follows by definitions.

Remark 1.9—In the above example, $\{0\}$ is prime, but not completely prime. This shows the necessity of σ-reflexivity in the hypothesis of Theorem 1.8.

As we have to consider in this paper, only σ-reflexive semigroups, in view of the above theorem, we do not make any distinction between the prime and completely prime ideals.

Proposition 1.10—Let S be a σ-reflexive semigroup and Q, a primary ideal. Then $\sqrt{Q}$ is prime.

PROOF : Let $ab \in \sqrt{Q}$, $a \notin \sqrt{Q}$. There exists $m \geqslant 1$, such that $(ab)^m \in Q$. If $ab = ba$, then $(ab)^m = a^m \,.\, b^m \in Q$; $a \notin \sqrt{Q} \Rightarrow a^m \notin Q \;\forall\; m$, and Q being primary, we get $(b^m)^p \in Q$ for some p; whence $b \in \sqrt{Q}$ and $\sqrt{Q}$ is prime. If, on the other hand, $ab \neq ba$, then since S is σ-reflexive, we have $ab = (ab)^t$ for some $t > 1$. If $m \leqslant t$, then $ab = (ab)^t = (ab)^m\,(ab)^{t-m} \in Q$, i.e., $ab \in Q$, $a \notin \sqrt{Q}$, whence $b \in Q$ and $b \in \sqrt{Q}$, again proving $\sqrt{Q}$ to be prime.

Next, if $m > t$, then $m = tq + r$ $(0 \leqslant r < t)$; $(ab)^m = (ab)^{qt}. (ab)^r = (ab)^q \,.\, (ab)^r = (ab)^{q+r}$, using $ab = (ab)^t$. If $q + r \geqslant t$, by repetition of the above procedure, we get ultimately $(ab)^m = (ab)^p \in Q$ and $p \leqslant t$. The argument in the previous paragraph is again applicable.

Definitions 1.11—Let S be a σ-reflexive semigroup. Then S is said to be primary if each of its ideals is primary. An ideal A of S is said to be semiprimary if $\sqrt{A}$ is prime. S is said to be a semiprimary semigroup if each of its ideals is semiprimary (Lal 1975). By Proposition 1.10, $\sqrt{Q}$ of a primary ideal Q is prime; therefore, a primary ideal is semiprimary and hence a primary semigroup is semiprimary. S is said to satisfy condition (*) (Gilmer 1962), if every semiprimary ideal of S is primary.

2. Semiprimary and Primary Semigroups

Remark 2.1—Every one-sided ideal of a σ-reflexive semigroup is clearly a two-sided ideal.

Lemma 2.2—Let S be a σ-reflexive semigroup and A any ideal of S. Then $\sqrt{A} = \cap P$, where intersection is over all prime ideals P of S, such that $A \subseteq P$.

Proof: Clearly $\sqrt{A} \subseteq \cap P$. For the reverse inclusion, suppose $x \notin \sqrt{A}$; then $T = \{x, x^2, x^3, \ldots\}$ is a subsemigroup of S, such that $T \cap A = \phi$. An application of Zorn's lemma yields an ideal $P \supseteq A$, $T \cap P = \phi$ and P is maximal with respect to this property. This P can be easily seen to be a prime ideal and it does not contain x. Hence, $\cap P \subseteq \sqrt{A}$.

Theorem 2.3—Let S be a σ-reflexive semigroup. Then S is semiprimary if prime ideals of S are totally ordered.

Proof: Let, first, prime ideals be totally ordered and A any ideal. Then by the above lemma, $\sqrt{A} = \cap P$. We prove that $\cap P$ is prime. Let $a \notin \cap P$ and $b \notin \cap P$. Then $a \notin P_1$ and $b \notin P_2$ for some primes P_1 and P_2. But $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$ To be definite, let $P_1 \subseteq P_2$. Then $b \notin P_1$, whence $ab \notin P_1$ and hence $ab \notin \cap P$, proving that $\cap P$ is prime, i.e., A is semiprimary and so is S. Conversely, let S be semiprimary and suppose there exists prime ideals P and P', such that $P \not\subseteq P'$ and $P' \not\subseteq P$. This gives $a \in P \setminus P'$ and $b \in P' \setminus P$, whence $ab \in P \cap P'$, $a \notin P \cap P'$, $b \in P \cap P'$, i.e., $P \cap P'$ is not a prime ideal. But $P \cap P' = \sqrt{P \cap P'}$, and as S is semiprimary, $P \cap P'$ is prime. This contradiction establishes that prime ideals of S are totally ordered.

Lemma 2.4—(Chacron and Thierrin 1972)—Every idempotent of a σ-reflexive semigroup is in its centre.

Corollary 2.5—Let S be a σ-reflexive semiprimary semigroup. Then its idempotents form a chain under natural ordering.

Proof: By 2.1, and 2.3, $\sqrt{eS} \subseteq \sqrt{fS}$ or $\sqrt{fS} \subseteq \sqrt{eS}$ for any two idempotents e and f. By the above lemma, $e = ef$ or $f = ef$.

Theorem 2.6—Let S be a regular σ-reflexive semigroup. Then the following statements are equivalent:

(i) Every ideal of S is prime.

(ii) S is a primary semigroup.

(iii) S is a semiprimary semigroup.

(iv) The idempotents form a chain.

(v) The ideals are totally ordered.

PROOF : (iv) $\Rightarrow$ (v). Let A and B be any two ideals, such that $B \subseteq\!\!\!\!|\;\; A$. Then there exists $b \in B \setminus A$. For any $a \in A$, $aS = eS$ and $bS = fS$, where e and f are some idempotents of S, by regularity of S. As $b \in B \setminus A$, we have $f \leqslant\!\!\!\!|\;\; e$ and since idempotents form a chain, $e \leqslant f$, whence $aS \subseteq bS$ and consequently $A \subseteq bS \subseteq B$, i.e., the ideals of S are totally ordered. (v) $\Rightarrow$ (i). Let A be any ideal of S and $xy \in A$. There exist idempotents e and f, such that $Sx = Se$ and $yS = fS$. Also, Se and fS are ideals. Assume for the sake of definiteness, $Se \subseteq fS$, whence $e = fe = ef = sx \,.\, yt \in A$. Hence, $x \in A$. Similarly, the other case proves that A is prime. The remaining implications follow by the previous results and definitions.

This theorem yields that if S is a regular (σ-reflexive) semigroup, then the two concepts of semiprimary and primary semigroups coincide. In general, these are two different concepts, as can be seen by Example 2 of Lal (1975).

Corollary 2.7—Let S be a σ-reflexive semiprimary semigroup. Then every ideal of S is prime if S is a regular semigroup.

PROOF : Let every ideal of S be prime. Then for any ideal A, A^2 is prime and hence $A^2 = A$. Thus, for any $x \in S$, $S^1 \,.\, x = (S^1.x)^2 = S^1.x^2$, since S is σ-reflexive. But then $x = tx^2$ for some $t \in S^1$. If $t = 1$, then x is idempotent and hence regular, and if $t \neq 1$, i.e., $t \in S$, then $x = tx^2 = (xt)^m.\, x$, i.e., x is again regular. Therefore, S is regular. The converse follows by the above theorem.

Combining this with Theorem 2.6, we get

Corollary 2.8—Let S be a σ-reflexive semigroup. Then every ideal of S is prime if S is regular and its idempotents form a chain.

Proposition 2.9—Let S be a σ-reflexive semigroup with identity. If every prime ideal is maximal, then S is a primary semigroup.

PROOF : By hypothesis, S has a unique proper maximal ideal M, which is also prime and any element not in M is a unit. For any ideal $A \neq S$ $\sqrt{A} = M$, as M is the only prime ideal by hypothesis. This means A is primary and so is S.

However, S to be primary does not imply that every prime ideal is maximal. The example in Remark 2.6 (Satyanarayana 1972) gives an illustration.

Lemma 2.10—A set P of a σ-reflexive semigroup S is a minimal prime ideal belonging to an ideal A of S if $S \setminus P$ is maximal subsemigroup not meeting A.

Lemma 2.11—Any prime ideal containing an ideal of a σ-reflexive semigroup contains a minimal prime ideal belonging to A.

McCoy (1962, pp. 103–107), though working in commutating rings, provides a proof which can be easily adapted for σ-reflexive semigroups for proving the above two lemmas.

Proposition 2.12—Let S be a left cancellative σ-reflexive semigroup satisfying condition (*). If Q is a primary ideal with $\sqrt{Q}$ a non-maximal prime ideal, then Q is prime.

PROOF: There exists an ideal A, such that $\sqrt{Q} \subset A \subset S$, by hypothesis. Choose $y \in A \setminus \sqrt{Q}$ and let $x \in \sqrt{Q}$. Then $\sqrt{Q \cup yxS^1} = \sqrt{Q}$. By condition (*) $Q \cup yxS^1$ is primary. For a $t \in S \setminus A$, $yxt \in Q \cup yxS^1$, $y \notin \sqrt{Q} \Rightarrow xt \in Q \cup yxS^1$. For $xt \in yxS^1$, we have $xt = yxt = (xy)^m \cdot t'$ $(t' \in S^1)$, whence by left cancellation, $t = (yx)^{m-1} \cdot yt' \in A$. This is not possible. Therefore, $xt \in Q$, $t \notin \sqrt{Q}$, and Q is primary, $x \in Q$. Hence, $Q = \sqrt{Q}$ and Q is prime.

Lemma 2.13—Let S be a left cancellative σ-reflexive semigroup. If there exist two non-commuting elements in S, then S is with identity and both these elements are units.

PROOF: Let $ab \neq ba$. Then $ab = (ab)^s$, where $s > 1$, whence $(ab)^{s-1}$ is an idempotent. But an idempotent in a left cancellative semigroup is a left identity of S. Also, this idempotent is in the centre of S and, therefore, it is an identity. Clearly $(ba)^{s-1}$ is an idempotent again and hence it equals the identity, whence a and b are both units.

Theorem 2.14—Let S be a left cancellative σ-reflexive semigroup satisfying condition (*). Then every proper prime ideal of S is maximal.

PROOF: Let P be a proper prime ideal. Let $p \in P$. By Lemma 2.11, there is a minimal prime ideal P_1 such that $pS^1 \subseteq P_1 \subseteq P$. We shall show that P_1 is maximal. Suppose it is not. Call $T = S \setminus P_1$ and $A = \{a \in S\colon at \in pS^1 \text{ for some } t \in T\}$. T is a subsemigroup and A is an ideal contained in P_1. Now take any $b_1 \in P_1$. Put $T_1 = \{b_1^r \cdot t : t \in T \text{ and } r \geqslant o\}$. For any two elements $b_1^r \cdot t$ and $b_1^s \cdot t'$, the product $b_1^r \cdot t \cdot b_1^s \cdot t'$ is in T, if $tb_1^s = b_1^s \cdot t$. Also, if $tb_1^s \neq b_1^s \cdot t$, then by Lemma 2.13, t and b_1^s are units and hence $P_1 = P = S$, which is a contradiction. Thus, T_1 is a subsemigroup of S, and it contains T properly. As P_1 is a minimal prime ideal belonging to the ideal pS^1, T is a maximal subsemigroup not meeting pS^1, and hence $T_1 \cap pS^1$ is nonempty, whence $b_1^r \cdot t \in pS^1$ for some $r \geqslant 1$ and $t \in T$, whence $b_1 \in \sqrt{A}$. Thus, $P_1 = \sqrt{A} = A$, by Proposition 2.12. Again, P_1 is a minimal prime ideal containing the ideal $p^2 \cdot S_1$. A repetition of the previous procedure will yield $P_1 = \sqrt{B} = B$, where $B = \{b \in S : bt \in p^2 \cdot S^1 \text{ for some } t \in T\}$. Thus, we have $A = B$. But $p \in A$, and, therefore, $p \in B$, whence $pt = p^2 \cdot x$ for some $t \in T$ and $x \in S^1$, whence by left cancellation, $t = px \in T \cap P_1$, which is not possible. Hence, P_1 is a maximal ideal and so is P.

Remark 2.15—Left cancellation cannot be dropped from the hypothesis of the above theorem.

Theorem 2.16—Let S be a left cancellative σ-reflexive semigroup with identity. Then the following statements are equivalent :

(i) S is a primary semigroup.

(ii) S satisfies condition (*).

(iii) Every proper prime ideal is maximal.

(iv) If a and b are nonunits in S, then $a^m = bx$ and $b^n = ay$ for some x, y in S and for some positive integers m and n.

PROOF : (iii) $\Rightarrow$ (iv) . If S is not a group, then as S is with identity, it has a unique maximal ideal M, which is also prime, and every nonunit will be contained in it. Thus, by Lemma 2.2, $\sqrt{aS} = \sqrt{bS} = M$, whence follows (iv). Other implications are immediate.

Theorem 2.17—Let S be a left cancellative σ-reflexive semigroup without identity. Then the following statements are equivalent :

(i) S is a primary semigroup.

(ii) S has no proper prime ideals.

(iii) If $a, b \in S$, then there exist integers m and n, such that $a^m = bx$ and $b^n = ay$ for some $x, y \in S$.

PROOF : (i) $\Rightarrow$ (ii) . Since prime ideals are maximal by 2.14 and they are totally ordered by 2.3, S has either no proper prime ideals or it has a unique proper prime ideal, which is maximal also. The latter alternative, we prove, is untenable. Suppose P is the unique proper prime ideal, which is also maximal. For an $a \notin P$, we have $P \cup aS^1 = P \cup a^2 . S^1 = S$, whence $a = a^2 . x$ for some $x \in S$. Now $a = a^2 . x = a . (xa)^m$. Thus, a is regular and $S \setminus P$ is a regular semigroup with exactly one idempotent e as its identity. Again, by left cancellation, $S \setminus P$ is a group. Clearly, $e \notin P$ and $\sqrt{Se} = S$, whence for any $x \in S$, $x^n = x^n . e$. By left cancellation, $x = xe$; and hence e is the identity of S, a contradiction. Therefore, S has no proper prime ideals. (ii) $\Rightarrow$ (iii). For any $a, b \in S$, $\sqrt{aS^1} = \sqrt{bS^1} = S$, whence follows (iii). (iii) $\Rightarrow$ (i) is obvious.

3. ARCHIMEDEAN SEMIGROUPS

Definition 3.1—Let S be a σ-reflexive semigroup. For any $a, b \in S$, we define: a divides b if there exists $x \in S^1$, such that $b = ax$. If for any pair of elements a, b of S, each divides some power of the other, then we call S to be an Archimedean semigroup.

Theorem 3.2—A σ-reflexive semigroup is Archimedean if it has no proper prime ideals.

Its proof is obvious.

Definition 3.3 (Petrich 1964)—A semigroup S is called weakly commutative if for any $a, b \in S$, we have $(ab)^k = xa = by$ for some $x, y \in S$ and a positive integer k. Clearly, a σ-reflexive semigroup is weakly commutative in view of Proposition 1.2.

Theorem 3.4—Every σ-reflexive semigroup is expressible as a semilattice of Archimedean subsemigroups.

This follows from Theorem 2 of Pondelicek (1975) in view of the fact that every σ-reflexive semigroup is weakly commutative.

Corollary 3.5—Let S be a σ-reflexive semigroup. Then every prime ideal in S is a union of Archimedean subsemigroups of S.

PROOF: By the above theorem, $S = \cup S_\alpha$, where S is an Archimedean subsemigroup of S. For a prime ideal P, let $\{S_\beta\}$ be a collection of subsemigroups from amongst the collection of Archimedean subsemigroups $\{S_\alpha\}$, such that $P \cap S_\beta$ is nonempty for each β. Then, $P \subseteq \cup S_\beta$. Also, $P \cap S_\beta$ is a prime ideal of S_β, and S_β, being Archimedean, has no proper prime ideal by Theorem 3.2, implying that $P \cap S_\beta = S_\beta$, i.e., $S_\beta \subseteq P$, whence $P = \cup S_\beta$.

Theorem 3.6—Let S be a σ-reflexive semigroup. Then the following statements are equivalent :

(i) Prime ideals are maximal and idempotents form a chain.

(ii) S is an Archimedean semigroup or there exists only one prime ideal P which is Archimedean and $S \setminus P$ is a group.

(iii) Each prime ideal of S is maximal and there are at most two idempotents in S.

PROOF : We work out a cyclic proof. First we prove (iii) $\Rightarrow$ (i). Let e and f be the only two idempotents. Since S is σ-reflexive, $ef = fe$ and it is an idempotent. Therefore, $ef = fe = e$ or $ef = fe = f$. Now we come to (i) $\Rightarrow$ (ii). As prime ideals of S are maximal, we have two possibilities : (*a*) There are no proper prime ideals in which case S is Archimedean by Theorem 3.2, and (*b*) S contains proper prime ideals. Let P be any one of them. Now for an $x \in S \setminus P$, $x^2 \in S \setminus P$, and P, being a maximal ideal of S, $P \cup x\ S^1 = P \cup x^2S^1 = S$, whence $x = x^2.y$, for some $y \in S^1$. Clearly, we can choose such a y in S. But then $xy = (yx)^n$ for some n. Hence, $x = x^2.y = x(xy) = x(yx)^n$ and x is regular i.e., $S \setminus P$ is regular, and hence contains idempotents. Let e and f be two idempotents in $S \setminus P$. This means $P \cup eS = P \cup fS = S$, whence $e = fe$ and $f = ef$. Hence, $e = f$. This means that $S \setminus P$ is a group. Similarly, for any other proper prime ideal Q of S, $S \setminus Q$ is a group. Let e_1 and e_2 be their respective identities. Suppose next that $e_1 \leqslant e_2$, as idempotents are given to form a chain, i.e., $e_1 = e_1 \,.\, e_2 = e_2 \,.\, e_1$, whence $e_2 \notin P$. This, however, means that $e_1 = e_2$ as $S \setminus P$ is a group. Now any $x \in S \setminus P$ has an inverse y, such that $e_1 = xy$, whence $x \in S \setminus Q$ and $S \setminus Q \supseteq S \setminus P$. By symmetry of this argument, we get $P = Q$ and we have a unique proper prime ideal P, with $S \setminus P$ to be a group. Also, $a, b \in P$ implies $\sqrt{aS^1} = \sqrt{bS^1} = P$, whence P is Archimedean. And lastly to (ii) $\Rightarrow$ (iii). In the first case, trivially every prime ideal is maximal. In the second case, let there exist a proper ideal A containing P (the unique proper prime ideal) properly. Thus, for $x \in A \setminus P$, there exists $y \in S \setminus P$, such that $xy = e$, where e is the identity of the group $S \setminus P$. This means $e \in A$. Now for any $b \in S \setminus A$, we have $b \in S \setminus P$ and hence $b = be \in A$, which is clearly not possible. Therefore the unique proper prime ideal P is maximal as well. For the second part in (iii), it suffices to show that P has at most one idempotent. Let there be two : e_1 and e_2. Then $\sqrt{e_1}\,.\,S = \sqrt{e_2}.S = P$, whence $e_1 = e_2$, as S is σ-reflexive.

Theorem 3.7—Let S be a σ-reflexive semigroup with no identity. Then S is a primary semigroup in which prime ideals are maximal iff S is Archimedean.

Proof : Let S be primary in which every prime is maximal. By 2.5, idempotents of S form a chain. Then by 3.6, S is Archimedean or there is a unique proper prime ideal P in S, such that $S \setminus P$ is a group and P, itself, is an Archimedean subsemigroup. We show that the second alternative is untenable. Let it hold. Then by hypothesis, the identity e of $S \setminus P$ cannot be the identity of S, which means for some x in P, $x \neq ex$. Clearly, $ex \in Sex$ and $x \notin Sex$. Also, no power of e is in Sex, whence the ideal Sex, is not primary, which is against the hypothesis. Thus, S is Archimedean. The converse follows by 3.2 and 2.2.

We now have two immediate corollaries.

Corollary 3.8—Let S be a σ-reflexive semigroup. Then the following are equivalent :

(i) S is an Archimedean semigroup.

(ii) S is a primary semigroup having no proper prime ideals.

(iii) S is a semiprimary semigroup and it has no proper prime ideals.

Corollary 3.9—Let S be a left cancellative σ-reflexive semigroup with no identity. Then the following are equivalent :

(i) S is primary.

(ii) S possesses no proper prime ideals.

(iii) S is Archimedean.

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