AN EXPANSION PROBLEM IN A DIRICHLET NORMED SPACE

by (MISS) P. PAIK, *Department of Mathematics, Govt. Engineering College, Jabalpur*

and

B. R. BHONSLE, *Department of Mathematics, Shivaji University, Kolhapur* 416004

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The D-norm for a function $u(r, \theta)$ in the domain $D\{0 < r < R, 0 \leqslant \theta \leqslant \pi/2\}$ has been defined as follows :

$$\| u \| = \left[\int_0^{\pi/2} \int_0^R r^{2\mu+2\nu-1} \left\{ r^2 \left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial \theta}\right)^2 \right\} \cos^{2\mu} \theta \sin^{2\nu} \theta \, dr \, d\theta \right]^{\frac{1}{2}},$$

$\mu > 0, \ \nu > 0.$

It will be said that $f(r, \theta)$ belongs to the D-normed space if

$$|| f(r, \theta) || < \infty.$$

In this paper, the expansion of $f(r, \theta)$ has been obtained in the D-normed space in terms of functions which are orthonormal, with respect to D-norm over the domain D.

1. INTRODUCTION

Recently, Lo (1972) considered the space of solutions of the following family of singular elliptic partial differential equations:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\mu}{x} \frac{\partial u}{\partial x} + \frac{2\nu}{y} \frac{\partial u}{\partial y} = 0$$

$$(\mu, \nu > 0, \quad x > 0, \quad y > 0). \qquad \ldots(1.1)$$

Equivalently, this equation can be written in the polar form as follows:

$$\frac{\partial^2 u}{\partial r^2} + (1 + 2\mu + 2\nu) \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

$$+ 2 (\nu \cot \theta - \mu \, tan \tan \theta) \frac{1}{r^2} \frac{\partial u}{\partial \theta} = 0 \qquad \ldots(1.1)$$

Let $u(r, \theta)$, $v(r, \theta)$ be two solutions of (1.1) in the domain $D\{0 < r < R, 0 < \theta < \pi/2\}$, then

$$E\{u, v\} = \int_0^R \int_0^{\pi/2} r^{2m+2\nu-1} \sin^{2\nu} \theta \cos^{2\mu} \theta \left[r^2 \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} + \frac{\partial u}{\partial \theta} \frac{\partial v}{\partial \theta} \right] d\theta \, dr$$

defines an inner product, and gives rise to the norm

$$\| u \|^2 = E\{u, u\} = \int_0^R \int_0^{\pi/2} r^{2\mu+2\nu-1} \sin^{2\nu}\theta \cos^{2\mu}\theta \left[r^2 \left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial \theta}\right)^2 \right] \times d\theta\, dr.$$

The space of all solutions of (1.1) in the domain D with the above norm and satisfying $\| u(r, \theta) \| < \infty$ will be called Dirichlet normed space or simply a D-normed space, which is, in fact, a Hilbert space.

Using the routine method of separating variables by assuming $u = R(r)\,\Theta(\theta)$ one is led to the following two sets of solutions of the resulting two ordinary differential equations:

$$R(r) = r^{2n}, \text{ and } \Theta(\theta) = P_n^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}(1 - 2\sin^2\theta)$$
$$(\nu > 0, \mu > 0, \quad n = 0, 1, 2, \ldots) \qquad \ldots(1.2)$$

where $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials. Thus, the solutions of (1.1) can be written as

$$u(r, \theta) = r^{2n} P_n^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}(\cos 2\theta), \quad n = 0, 1, 2, \ldots.$$

In fact Lo (1972) has shown that

$$u_n(r, \theta) = (C_{n,\nu-\frac{1}{2},\mu-\frac{1}{2}})^{1/2} r^{2n} R^{-2n-\mu-\nu} P_n^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}(\cos 2\theta) \qquad \ldots(1.3)$$

where

$$C_{n,\alpha,\beta} = \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\,\Gamma(n)\,\Gamma(n+\alpha+\beta+1)} \qquad \ldots(1.4)$$

is an orthonormal set with respect to the D-norm over the domain D.

A careful analysis of (1.2) shows that r^{2n} is a complete set in $L^2(I)$ (I, any interval) and so also $\{P^{(\alpha,\beta)}(x)\}$ in $L^2[-1, 1]$ (i.e., in $L^2[0, \pi/2]$ in polar form). Hence, using quite elementary arguments, one can easily conclude that the set $\{u_n(r, \theta)\}$ forms a complete orthonormal set in the D-normed space. It may be mentioned that by adopting the approach of Bergmann and Schiffer (1953, pp. 258–97), one could also arrive at the same conclusion. Our conclusion is also implicit in the considerations of Davis (1956, pp. 207–25) whose Hilbert space $L^2(B)$ contains D-normed spaces as particular cases.

Now the set (1.3) of solutions is complete; hence by the well-known Hilbert space arguments one concludes that for any element $f(r, \theta)$ in the D-normed space we have

$$f(r, \theta) = \sum_{n=0}^{\infty} a_n u_n(r, \theta) \qquad \ldots(1.5)$$

where $a_n = E\{u_n, f)\}$. In a nutshell, we are now permitted to interchange summation and integration as well as termwise integration and differentiation.

In this note, we are mainly concerned with the expansion problem in a D-normed space.

2. Results Required

In this section, we enlist those results which are needed in our analysis. From Sneddon (1961, p. 37) we have

$$ {}_1F_1(\alpha;\ \gamma;\ x) = \frac{1}{B(\alpha, \nu - \alpha)} \int_0^1 (1-t)^{\nu-\alpha-1}\, t^{\alpha-1}\, e^{xt}\, dt. $$

Putting $t = r/R$ and $\nu = \alpha + 1$, we have

$$ \int_0^R r^{\alpha-1}\, e^{(x/R)r}\, dr = R^{\alpha}\, B(\alpha, 1)\, {}_1F_1(\alpha; \alpha+1; x) \qquad \ldots(2.1) $$

From Erdelyi *et al.* [p. 284 (1)], we have

$$ \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\sigma}\, P_n^{(\alpha,\beta)}(x)\, dx $$

$$ = \frac{2^{\sigma+\alpha+1}\,\Gamma(\sigma+1)\,\Gamma(\alpha+n+1)\,\Gamma(\sigma-\beta+1)}{\Gamma(\sigma-\beta-n+1)\,\Gamma(\sigma+\alpha+n+2)\,n!}\ . \qquad \ldots(2.2) $$

3. Expansion of $f(r, \theta)$ in the D-Normed Space

Let us expand $f(r, \theta)$, which is in the D-normed space. On account of our analysis and (1.5), if we write

$$ f(r, \theta) = \sum_{n=0}^{\infty} a_n\, u_n(r, \theta), \qquad \ldots(3.1) $$

then

$$ a = E\{u_n, f\} = I_1 + I_2, $$

where

$$ I_1 = \int_0^{\pi/2}\int_0^R r^{2\mu+2\nu+1} \frac{\partial f}{\partial r}\frac{\partial u_n}{\partial r} \cos^{2\mu}\theta \sin^{2\nu}\theta\, dr\, d\theta, \qquad \ldots(3.2) $$

and

$$ I_2 = \int_0^{\pi/2}\int_0^R r^{2\mu+2\nu-1} \frac{\partial f}{\partial \theta}\frac{\partial u_n}{\partial \theta} \cos^{2\mu}\theta \sin^{2\nu}\theta\, dr\, d\theta \qquad \ldots(3.3) $$

From (1.3), differentiating $u(r, \theta)$ with respect to r and θ and putting

$$ A = (C_{n,\,\nu-\frac{1}{2},\,\mu-\frac{1}{2}})^{-\frac{1}{2}}\, R^{-2n-\mu-\nu} \qquad \ldots(3.4) $$

we get

$$I_1 = 2nA \int_0^{\pi/2} \int_0^R r^{2\mu+2\nu+n} \frac{\partial f}{\partial r} P_n^{(\nu+\frac{1}{2},\, \mu+\frac{1}{2})} (\cos 2\,\theta) \cos^{2\mu}\theta \sin^{2\nu}\theta \, dr\, d\theta \qquad ...(3.5)$$

$$I_2 = -2(\mu+\nu+n) A \int_0^{\pi/2} \int_0^R r^{2\mu+2\nu+2n-1} \frac{\partial f}{\partial \theta}$$

$$\times P_n^{(\nu+\frac{1}{2},\, \mu+\frac{1}{2})}(\cos 2\theta) \cos^{2(\mu+\frac{1}{2})}\theta \sin^{2(\nu+\frac{1}{2})}\theta \; dr\, d\theta. \qquad ...(3.6)$$

4. Example

In particular, let

$$f(r,\theta) = r^{\alpha-1} e^{(x/R)r} (1+\cos 2\theta)^{\sigma-\mu-\frac{1}{2}}. \qquad ...(4.1)$$

First of all we show that $f(r,\theta)$ belongs to the D-normed space. For this, we have to show that $\| f(r,\theta) \| < \infty$. We observe that

$$\| f(r,\theta) \|^2 = \int_0^{\pi/2} \int_0^R r^{2\mu+2\nu-1} \left\{ r^2 \left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial \theta}\right)^2 \right\}$$

$$\times \cos^{2\mu}\theta \sin^{2\nu} dr d\theta, \quad \mu, \nu > 0$$

$$= 2^{2(\sigma-\mu)} R^{2\mu+2\nu+2\alpha-2} [\{(\alpha-1)^2 B(2\mu+2\nu+2\alpha-2, 1)$$

$$\times {}_1F_1 [2\mu+2\nu+2\alpha-2;\, 2\mu+2\nu+2\alpha-1;\, 2x]$$

$$+ 2(\alpha-1) x B(2\mu+2\nu+2\alpha-1, 1)$$

$$\times {}_1F_1 [2\mu+2\nu+2\alpha;\, 2\mu+2\nu+2\alpha+1;\, 2x]\} \tfrac{1}{4} B(2\sigma-\mu-\tfrac{1}{2}, \nu+\tfrac{1}{2})$$

$$+ (\sigma-\mu-\tfrac{1}{2})^2 B(2\mu+2\nu+2\alpha-2, 1)$$

$$\times {}_1F_1 [2\mu+2\nu+2\alpha-2;\, 2\mu+2\nu+2\alpha-1;\, 2x]$$

$$\times B(2\sigma-\mu-3/2, \nu+3/2)]. \qquad ...(4.2)$$

Hence, we conclude that $f(r,\theta)$ is finite.

To evaluate a_n, we differentiate (4.1) with respect to r and θ, and substitute in (3.5) and (3.6) respectively to obtain

$$I_1 = 2nA \int_0^{\pi/2} \int_0^R r^{2(\mu+\nu+n)} (\alpha-1) r^{\alpha-2} e^{(x/R)r}$$

$$\times (1+\cos 2\theta)^{\sigma-\mu-\frac{1}{2}} P_n^{(\nu-\frac{1}{2},\, \mu-\frac{1}{2})} (\cos 2\theta) \sin^{2\nu}\theta \cos^{2\mu}\theta \, dr\, d\theta$$

$$+ 2nA - \frac{x}{R} \int_0^{\pi/2} \int_0^R r^{2\mu+2\nu+2n-\alpha-1} e^{(x/R)r}$$

$$\times (1+\cos 2\theta)^{\sigma-\mu-\frac{1}{2}} P_n^{(\nu-\frac{1}{2},\, \mu-\frac{1}{2})} (\cos 2\theta) \sin^{2\nu}\theta \cos^{2\mu}\theta \, dr\, d\theta$$

$$I_1 = \Big[nA(\alpha - 1)\, 2^{\sigma-\mu-\frac{1}{2}} \int_0^R r^{2(\mu+\nu+n)+\alpha-2}\, e^{(x/R)r}\, dr$$

$$+ 2^{\sigma-\mu+\frac{1}{2}} \frac{nAx}{R} \int_0^R r^{2(\mu+\nu+n)+\alpha-1}\, e^{(x/R)r}\, dr \Big]$$

$$\times \int_0^{\pi/2} \sin^{2\nu}\theta \cos^{2\mu}\theta\, P_n^{(\nu-\frac{1}{2},\, \mu-\frac{1}{2})}(\cos 2\theta)\, d\theta \qquad \ldots(4.3)$$

Using the results (2.1) and (2.2), we obtain

$$I_1 = 2^{\sigma-\mu-\frac{1}{2}} A R^{2\mu+2\nu+2\alpha-1} \{ n(\alpha-1) B(2\mu + 2\nu + 2n + \alpha - 1, 1)$$

$$\times {}_1F_1 [2\mu + 2\nu + 2n + \alpha - 1;\; 2\mu + 2\nu + 2n + \alpha;\; x]$$

$$+ xB(2\mu + 2\nu + 2n + \alpha, 1)$$

$$\times {}_1F_1 [2\mu + 2\nu + 2n + \alpha;\, 2\mu + 2\nu + 2n + \alpha + 1, x]\}$$

$$\times \frac{\Gamma(\sigma)\Gamma(\nu + n + \frac{1}{2})\; \Gamma(\sigma - m + \frac{1}{2})}{2\Gamma(\sigma - \mu - n + \frac{1}{2})\; \Gamma(\sigma + \nu + n + \frac{1}{2})\, n!} \quad . \qquad \ldots(4.4)$$

We also have

$$I_2 = -2(\mu + 2\nu + n) A \int_0^{\pi/2} \int_0^R r^{2\mu-2\nu+2n+\alpha-1} e^{(x/R)r}$$

$$\times (\sigma - \mu - \tfrac{1}{2})(2\cos^2\theta)^{\sigma-\mu-3/2} P_{k-1}^{(\nu+\frac{1}{2},\, \mu+\frac{1}{2})}(\cos 2\theta)$$

$$\times \sin^{2\nu+1}\theta \cos^{2\mu+1}\theta (-4\sin\theta\cos\theta)\, dr\, d\theta \qquad \ldots(4.5)$$

Using the results from (2.1) and (2.2), we get

$$I_2 = 2^{\sigma-\mu+\frac{1}{2}} A(\nu + \mu + n)(\sigma - \mu - \tfrac{1}{2}) R^{2\mu+2\nu+2n+\alpha-1}$$

$$\times B(2\mu + 2\nu + 2n + \alpha - 1, 1)$$

$$\times {}_1F_1 [2\mu + 2\nu + 2n + \alpha - 1; 2\mu + 2\nu + 2n + \alpha; x]$$

$$\times \frac{\Gamma(\sigma)\Gamma(\nu + n + \frac{1}{2})\Gamma(\sigma - \mu - \frac{1}{2})}{(n-1)!\,\Gamma(\sigma - \mu - n + \frac{1}{2})\Gamma(\nu + \sigma + n + \frac{3}{2})}$$

where A is given by (3.4). Adding (4.4) and (4.5), we get the value of a_n.

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