A SET PARTITIONING PROBLEM WITH LINEAR FRACTIONAL OBJECTIVE FUNCTION

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This paper develops a technique for finding an optimal partition of the set patitioning problem with linear fractional objective function. The technique evolved is an enumerative one. It involves evaluation of extreme points of the linear fractional programming problem, derived from the partitioning problem, in a given sequence. An example is included to illustrate the method.

Introduction

The set partition problem with linear objective function is

$$\text{Minimize } Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_j = 1 \qquad i = 1, 2, \ldots, m$$

$$x_j = 0 \text{ or } 1 \qquad j = 1, 2, ,, \ldots, n$$

where $A = [a_{ij}]$ is an $m \times n$ matrix of zeros and ones $c_j \geqslant 0$ for all j.

This paper is an extension of the above problem in the sense that here the objective function would be a linear fractional function. The technique developed is also different from the techniques reported in literature (Arabeyre *et al.* 1969, Garfunkel and Nemhauser 1969, Marsten 1974). The present technique involves the evaluation of extreme point solutions of a linear fractional programming problem till the optimal partition is obtained.

Mathematical Formulation

Consider a set $I = [1, 2, \ldots, m]$ and a set $P = [P_1, P_2, \ldots, P_n]$, where $P_j \subseteq I, j \in J = [1, 2, \ldots, n]$. A subset $J^* \subseteq J$ defines a cover of I if

$$\bigcup_{j \in J^*} P_j = I.$$

If, in addition, for all $j, k \in J^*$, $j \neq k \Rightarrow P_j \cap P_k = \phi$, then J^* defines a partition of I.

The mathematical model of our problem is :

Problem I

$$\text{Minimize } Z = \frac{\sum_{j=1}^{n} c_j x_j}{\sum_{j=1}^{n} d_j x_j + \beta}$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij}\, x_j = 1, \quad i = 1, 2, \ldots, m \qquad \ldots(1)$$

$$x_j = 0 \quad \text{or} \quad 1, \quad j = 1, 2, \ldots, n \qquad \ldots(2)$$

where

$x_j = 1$ if j is in the paritition

$= 0$ otherwise

$a_{ij} = 1$ if $i \in P_j$

$= 0$ otherwise.

The linear fractional programming problem obtained by replacing (2) with $x_j \geqslant 0$ in Problem I is

Problem II

$$\text{Minimize } Z = \frac{\sum_{j=1}^{n} c_j x_j}{\sum_{j=1}^{n} d_j x_j + \beta}$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij}\, x_j = 1, \qquad i = 1, 2, \ldots, m$$

$$x_j \geqslant 0, \qquad j = 1, 2, \ldots, n$$

It is assumed that the solution set of Problem II is regular and $\sum_{j=1}^{n} d_j x_j + \beta$ is strictly positive over this set.

Application

Partitioning problems have a number of applications. Some of them are 'air line crew scheduling', 'truck routing', 'political districting', 'information retrieval', etc.

We now present a simple transportation problem of a city to illustrate that set partitioning problems with linear fractional objective function do occur in practice.

Let $I = [1, 2, \ldots, m]$ be the set of m locations.

$J = [1, 2, \ldots, n]$ be the set of n routes by which the buses can go.

c_j = expenditure in terms of petrol when the bus trends by the jth route.

d_j = the amount that the owner earns by the jth route.

β = the fixed amount that the owner receives (it will be negative when the owner pays a fixed amount).

Define a matrix $A = [a_{ij}]$ as

$a_{ij} = 1$ if ith location is covered by the jth route.

$= 0$ otherwise.

Define a variable x_j as

$x_j = 1$, if the bus follows the jth route.

$= 0$ otherwise.

P_j = Set of locations covered by the jth route.

Clearly, $P_j \subseteq I$.

A subset J^* of J covers I, i.e., covers all the locations if

$$\bigcup_{j \in J^*} P_j = I.$$

Further, if each location is covered exactly once by the routes belonging to J^*, i.e., if for all $j, k \in J^*, j \neq k$ $P_j \cap P_k = \phi$, then J^* is known as a partition.

Our problem is to find a partition J^* which covers all the locations and minimizes the ratio

$$\sum_{j \in J^*} c_j / \left(\sum_{j \in J^*} d_j + \beta \right).$$

As $x_j = 1$ if the bus follows the jth route, i.e., if $j \in J^*$

$= 0$ otherwise.

Therefore, our problem becomes

$$\text{Minimize } Z = \frac{\sum_{j=1}^{n} c_j x_j}{\sum_{j=1}^{n} d_j x_j + \beta}$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_j = 1, \qquad i = 1, 2, \ldots, m$$

$$x_j = 0, 1, \qquad j = 1, 2, \ldots, n.$$

THEORETICAL DEVELOPMENT

Definition

Partition solution—A solution which satisfies (1) and (2) is called a partition solution and if it minimizes Z, it is called an Optimal partition solution.

Theorem—Let $J' = \{j/x_j = 1\}$ be any partition solution, then $X = [x_j]$ is an extreme point of the set of feasible solutions of Problem II.

PROOF: Since J' is a partition, $P_j \cap P_k = \phi \ \forall\ j, k \in J'$ and $j \neq k$, which implies that the columns corresponding to members of J' have no positive entries in common. Therefore, the columns corresponding to the members of J' are linearly independent. Hence, they form a basic feasible solution. Therefore, X is an extreme point of the set of feasible solutions to Problem II.

Algorithm

The procedure for solving Problem I consists of the following steps:

Step I: Solve Problem II (for solving linear fractional functional programming refer to Martos 1960, Charnes and Cooper 1962, Kanti Swarup 1965).

Let its optimal basic feasible solution be

$$X_1 = [x_j].$$

If the variables in X_1 are such that

$$x_j = 0 \text{ or } 1 \ \forall\ j$$

then an optimal partition is reached which is

$$J' = [j/x_j = 1]$$

and the minimum value of Z is

$$\frac{\sum\limits_{j \in J'} c_j}{\sum\limits_{j \in J'} d_j + \beta}$$

if the variables in X_1 are not zero-one, go to Step II.

Step II. Find the ith best extreme point solution X_i starting from $i = 2$ of Problem II. (For the method of finding ith best extreme point refer to Kirby *et al.* 1972, Murty 1968, Puri and Kanti Swarup 1965). If the solution X_i is zero-one, then terminate, otherwise go to Step III.

Step III: Repeat Step II for next higher values of i, i.e., $i + 1$, $i + 2, \ldots$ till the optimal partition is obtained.

Convergence

The procedure is convergent because of the following facts:

(i) It involves movements from one extreme point to another extreme point of problem II, which are always finite in number.

(ii) None of the extreme points is repeated, as the value of the objective function is increased at each stage.

Example—Find an optimal partition solution for the following partitioning problem with linear fractional functional as its objective function:

$$\text{Minimize } Z = \frac{2x_1 + 9x_2 + 8x_3 + 6x_4 + 8x_5 + 7x_6 + 5x_7}{x_1 + 4x_2 + x_3 + 2x_4 + x_5 + 5x_6 + 2x_7 + 2}$$

subject to

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_3 + x_4 + x_5 &= 1 \\ x_5 + x_6 + x_7 &= 1 \\ x_2 + x_4 + x_6 &= 1 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7 &= 0, 1 \end{aligned}$$

Solution

Here, $I = [1, 2, 3, 4]$, $J = [1, 2, 3, 4, 5, 6, 7]$

$P_1 = [1], P_2 [1, 4], \quad P_3 = [2], \quad P_4 = [2, 4]$

$P_5 = [2, 3], P_6 = [3, 4], P_7 = [3].$

The linear fractional functional programming problem associated with the above problem is:

$$\text{Minimize } Z = \frac{2x_1 + 9x_2 + 8x_3 + 6x_4 + 8x_5 + 7x_6 + 5x_7}{x_1 + 4x_2 + x_3 + 2x_4 + x_5 + 5x_6 + 2x_7 + 2}$$

subject to

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_3 + x_4 + x_5 &= 1 \\ x_5 + x_6 + x_7 &= 1 \\ x_2 + x_4 + x_6 &= 1 \\ x_j \geqslant 0 & \qquad j = 1, 2, \ldots, 7. \end{aligned}$$

Step I: Solve this problem, the optimal solution is $X_1 = [1, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0]$ and the corresponding simplex table is

		$c_j \rightarrow$		2	9	8	6	8	7	5
		$d_j \rightarrow$		1	4	1	2	1	5	2
$c_B \downarrow$	$d_B \downarrow$	Vectors in basis $\downarrow$	$X_1 \downarrow$	f_1	f_2	f_3	f_4	f_5	f_6	f_7
2	1	x_1	1	1	1	0	0	0	0	0
6	2	x_4	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$
8	1	x_5	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	1	0	$\frac{1}{2}$
7	5	x_6	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	1	$\frac{1}{2}$
$z^{(1)} = \frac{25}{2}$	$z^{(2)} = 7$		$Z = \frac{25}{14}$							
		$c_j - z_j^{(1)} \rightarrow$		—	9/2	9/2	—	—	—	$\frac{1}{2}$
		$d_j - z_j^{(2)} \rightarrow$		—	0	2	—	—	—	0
		$\Delta_j \rightarrow$		—	$\frac{63}{2}$	$\frac{13}{2}$	—	—	—	7/2

where $z^{(1)} = c_B X_1$ and $z^{(2)} = d_B X_1 + 2$ and $Z = \frac{z^{(1)}}{z^{(2)}}$ $f_1, f_2, f_3, f_4, f_5, f_6, f_7$, correspond to the vector associated with the variables.

As X_1 is not a zero-one solution, go to Step II to find the second best extreme point solution.

Step II: The values of the objective function at extreme points adjacent to X_1 are:

(i) 17/7, when f_1 enters and f_2 departs,

(ii) 17/9, when f_2 enters and f_3 departs,

(iii) 13/7, when f_7 enters and f_5 departs.

Out of these values, the minimum is 13/7, which is obtained by entering f_7 and departing f_5.

Therefore, the second best value is 13/7 and the extreme point solution yielding this value is

$X_2 = (1,0,0,1,0,0,1)$ with the corresponding simplex table given below:

		$c_j \rightarrow$		2	9	8	6	8	7	5
		$d_j \rightarrow$		1	4	1	2	1	5	2
c_B ↓	d_B ↓	Vectors in basis ↓	X_2 ↓	f_1	f_2	f_3	f_4	f_5	f_6	f_7
2	1	x_1	1	1	1	0	0	0	0	0
6	2	x_4	1	0	0	1	1	1	0	0
5	2	x_7	1	0	−1	1	0	2	0	1
7	5	x_6	0	0	1	−1	0	−1	1	0
$z^{(1)} = 13$	$z^{(2)} = 7$	$Z = \frac{13}{7}$								
		$c_j - z_j^{(1)} \rightarrow$		—	5	4	—	−1	—	—
		$d_j - z_j^{(2)} \rightarrow$		—	0	2	—	0	—	—
		$\Delta_j \rightarrow$		—	35	2	—	−7	—	—

As X_2 is a zero-one solution, the process terminates and the optimal partition is given by

$$J' = [1, 4, 7]$$

and the value of the objective function is $Z = 13/7$.

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