COEFFICIENT ESTIMATES FOR CERTAIN CLASSES OF MEROMORPHIC FUNCTIONS

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In this paper, we have obtained the coefficient estimate for the functions $f(z) = z^{-p} + a_0 z^{-p+1} + a_1 z^{-p+2} + \cdots + a_{n+p-1} z^n + \cdots$ which are regular in $E(0 < |z| < 1)$ and maps E onto a domain whose complement's boundary is of bounded radial rotation. We have also obtained a similar result for the function $f(z)$ which is regular in E and maps E onto a domain whose complement's boundary is of bounded boundary rotation.

§1. Let $BV[0, 2\pi]$ be the class of all real valued functions and of bounded variation in $[0, 2\pi]$. We denote by $MC_p^k(\alpha)$, the set of functions $f(z) = z^{-p} + a_0 z^{-p+1} + \ldots + a_{p+n-1} z^n + \ldots$ that are regular in $E(0 < |z| < 1)$ and have an integral representation

$$f'(z) = -\frac{p}{z^{p+1}} \exp\left\{\frac{(p-\alpha)}{\pi p} \int_0^{2\pi} \log(1 - ze^{-it})\, d\mu(t)\right\} \qquad \ldots(1.1)$$

where p is a positive integer with $0 \leqslant \alpha < p$ and $\mu(t) \in BV[0, 2\pi]$ having the following properties

$$\int_0^{2\pi} d\mu(t) = 2\pi p, \qquad \int_0^{2\pi} |d\mu(t)| \leqslant kp\pi. \qquad \ldots(1.2)$$

Further, we denote $MS_p^{*k}(\alpha)$, the class of all functions $f(z)$ which are regular in E and have an integral representation .

$$f(z) = \frac{1}{z^p} \exp\left\{\frac{(p-\alpha)}{\pi p} \int_0^{2\pi} \log(1 - z\, e^{-it})\, d\mu(t)\right\} \qquad \ldots(1.3)$$

where $\mu(t) \in BV[0, 2\pi]$ and satisfied the conditions (1.2).

One can easily see that if $f(z) \in MC_p^k(\alpha)$ or $MS_p^{*k}(\alpha)$, then $f(z)$ maps E onto a domain whose complement's boundary is of bounded boundary rotation or bounded radial rotation respectively.

§2. In this paper, we have solved some coefficient problems for the classes $MS_p^{*k}(\alpha)$ and $MC_p^k(\alpha)$. Before starting with actual theorems, we need the following lemmas.

Lemma 2.1—$f(z) \in MS_p^{*k}(\alpha)$, if and only if, there exists $p_1(z)$ and $p_2(z) \in MS_1^{*2}(0)$ the class of all functions which are regular and starlike in E, such that

$$f(z) = \frac{1}{z^p} \frac{(zp_1(z))^{(k+2)(p-\alpha)/4}}{(zp_2(z))^{(k-2)(p-\alpha)/4}}. \qquad \ldots(2.1)$$

Lemma 2.2—$f(z) \in MC_p^k(\alpha)$, if and only if, there exists $p_1(z)$ and $p_2(z) \in MS_1^{*2}(0)$, the class of all functions which are regular and starlike in E, such that

$$f'(z) = -\frac{p}{z^{p+1}} \frac{(zp_1(z))^{(k+2)(p-\alpha)/4}}{(zp_2(z))^{(k-2)(p-\alpha)/4}}. \qquad \ldots(2.2)$$

Lemma 2.3—Let $p(z) \in MS_1^{*2}(0)$ and $\beta > 0$ let $g(z) = (zp(z))^{\beta} = \sum_{n=0}^{\infty} b_n z^n$. Then

$$|b_n| \leqslant \frac{\Gamma(2\beta + 1)}{\Gamma(n + 1)\Gamma(2\beta - n + 1)}, \quad 0 \leqslant n \leqslant \beta + 1. \qquad \ldots(2.3)$$

Lemma 2.4—Let $p(z) \in MS_1^{*2}(0)$ and let $\alpha > 0$. Let $g(z) = (zp(z))^{-\alpha} = \sum_{n=0}^{\infty} b_n z^n$. Then

$$|b_n| \leqslant \frac{\Gamma(n + 2\alpha)}{\Gamma(n + 1)\Gamma(2\alpha)}. \qquad \ldots(2.4)$$

Proofs of these lemmas are omitted, as they are parallel to the results of Paatero (1931) and Noonan (1971).

Theorem 2.1—Let $f(z) = z^{-p} + a_0 z^{-p+1} + \ldots + a_{p-1} + a_p z + \ldots + a_{p+n-1} z^n + \ldots$ be a member of $MS_p^{*k}(\alpha)$, and let $F_k(z) = z^{-p} + A_0 z^{-p+1} + \ldots + A_{p+n-1} z^n + \ldots$ be given by

$$F_k(z) = \frac{1}{z^p} \frac{(1 + z)^{(k+2)(p-\alpha)/2}}{(1 - z)^{(k-2)(p-\alpha)/2}}.$$

Then for $n \leqslant [(k + 2)(p - \alpha)/4 - (p - 1)]$, we have $|a_{p+1-1}| \leqslant |A_{p+n-1}|$. This result is sharp.

PROOF: Since $f(z) \in MS_p^{*k}(\alpha)$, from Lemma 2.1, we have

$$f(z) = \frac{1}{z^p} \frac{(zp_1(z))^{(k+2)(p-\alpha)/4}}{(zp_2(z))^{(k-2)(p-\alpha)/4}}$$

where $p_1(z)$, $p_2(z) \in MS_1^{*2}(0)$. Let

$$[zp_1(z)]^{(k+2)(p-\alpha)/4} = \sum_{j=0}^{\infty} c_j z^j$$

and

$$[zp_2(z)]^{-(k-2)(p-\alpha)/4} = \sum_{j=0}^{\infty} b_j z^j.$$

Therefore,

$$f(z) = z^{-p} + a_0 z^{-p+1} + \dots + a_{p+n-1} z^n + \dots$$

$$= z^{-p} \Big(\sum_{j=0}^{\infty} c_j z^j \Big) \Big(\sum_{h=0}^{\infty} b_h z^h \Big). \qquad \dots(2.5)$$

Comparing the coefficient of z^n in (2.5), we have

$$a_{p+n-1} = c_0 b_{p+n} + c_1 b_{p+n-1} + \dots + c_{p+n-1} b_1 + c_{p+n} b_0 \qquad \dots(2.6)$$

From Lemma 2.3 with $\beta = (k+2)(p-\alpha)/4$, we have

$$|c_j| \leqslant \frac{\Gamma\left\{\frac{k+2}{2}(p-\alpha)+1\right\}}{\Gamma(j+1)\Gamma\left\{\frac{k+2}{2}(p-\alpha)-j+1\right\}},\ 0 \leqslant j \leqslant \frac{k+2}{4}(p-\alpha)+1,$$

and from Lemma 2.4 with $\alpha = (k-2)(p-\alpha)/4$, we have

$$|b_m| \leqslant \frac{\Gamma\left\{m + \frac{k-2}{2}(p-\alpha)\right\}}{\Gamma(m+1)\Gamma\left\{\frac{k-2}{2}(p-\alpha)\right\}}.$$

Hence, from (2.6)

$$|a_{p+n-1}| \leqslant \sum_{j=0}^{p+n} |c_j|\,|b_{p+n-j}|$$

$$\leqslant \sum_{j=0}^{p+n} \frac{\Gamma\left\{\frac{k+2}{2}(p-\alpha)+1)\right\}}{\Gamma(j+1)\Gamma\left\{\frac{k+2}{2}(p-\alpha)-j+1\right\}} \times \frac{\Gamma\left\{n-j+p+\frac{k-2}{2}(p-\alpha)\right\}}{\Gamma(n-j+p+1)\Gamma\left\{\frac{k-2}{2}(p-\alpha)\right\}} \qquad \dots(2.7)$$

with $n \leqslant (k+2)(p-\alpha)/4 - (p-1)$.

Now let us examine

$$F_k(z) = \frac{1}{z^p}\,\frac{(1+z)^{(k+2)(p-\alpha)/2}}{(1-z)^{(k-2)(p-\alpha)/2}}.$$

Let

$$(1 + z)^{(k+2)(p-\alpha)/2} = \sum_{n=0}^{\infty} C_n z^n$$

and

$$(1 - z)^{-(k-2)(p-\alpha)/2} = \sum_{n=0}^{\infty} B_n z^n.$$

Then

$$C_n = \frac{\Gamma\left\{\frac{k+2}{2}(p-\alpha)+1\right\}}{\Gamma(n+1)\Gamma\left\{\frac{k+2}{2}(p-\alpha)-n+1\right\}}, \quad n \leqslant \frac{k+2}{2}(p-\alpha)+1$$

and

$$B_n = \frac{\Gamma\left\{n + \frac{k-2}{2}(p-\alpha)\right\}}{\Gamma(n+1)\Gamma\left\{\frac{k-2}{2}(p-\alpha)\right\}}.$$

Hence, we see that

$$A_{p+n-1} = \sum_{j=0}^{p+n} \frac{\Gamma\left\{\frac{k+2}{2}(p-\alpha)+1\right\}}{\Gamma(j+1)\Gamma\left\{\frac{k+2}{2}(p-\alpha)-j+1\right\}} \times \frac{\Gamma\left\{n-j+p+\frac{k-2}{2}(p-\alpha)\right\}}{\Gamma\{n-j+p+1\}\Gamma\left\{\frac{k-2}{2}(p-\alpha)\right\}} \qquad \ldots(2.8)$$

From (2.7) and (2.8), we have

$$|a_{p+n-1}| < |A_{p+n-1}|, \quad n \leqslant [(k+2)(p-\alpha)/4 - (p-1)].$$

Since $F_k(z) \in MS_p^{*k}(\alpha)$, the result is also sharp.

On similar lines as in the proof of Theorem 2.1, we have the following coefficient problem for the class $MC_p^k(\alpha)$.

Theorem 2.2—Let $f(z) = z^{-p} + a_0 z^{-p+1} + a_1 z^{-p+2} + \ldots + a_{p+n-1} z^n + \ldots \in MC_p^k(\alpha)$ and let $F'_k(z) = -pz^{-p-1} - (p-1)z^{-p}A_0 + \ldots + nA_{p+n-1}z^{n-1} + \ldots$ be given by

$$F'_k(z) = -\frac{p}{z^{p+1}} \frac{(1+z)^{(k+2)(p-\alpha)/2}}{(1-z)^{(k-2)(p-\alpha)/2}}$$

Then for $n \leqslant [(k+2)(p-\alpha)/4-(p-1)]$, we have $|a_{p+n-1}| \leqslant |A_{p+n-1}|$ except for a_{p-1}. This result is sharp.

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