ON THE ABSOLUTE NÖRLUND SUMMABILITY FACTORS OF INFINITE SERIES

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(*Received* 16 *June* 1976)

In this paper, the special case of a result of Ahmad [*Math. Z.*, **76** (1961), 295–310] has been generalized to absolute Nörlund summability factors of infinite series, in the same manner as Ahmad and Khan [*Indian J. Math.*, **16** (1974), 137–56] did to the results of Chow [*J. Lond. math. Soc.*, **29** (1954), 459–76, Theorem 2].

1.1. DEFINITIONS AND NOTATIONS

Let Σa_n be a given infinite series with the sequence of partial sums, $\{s_n\}$, and let s_n^α and t_n^α denote the nth Cesàro-means of order α ($\alpha > -1$), of the sequences $\{s_n\}$ and $\{na_n\}$ respectively, defined by:

$$s_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu ; \quad t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} \nu a_\nu , \qquad ...(1.1.1)$$

where

$$\sum_{n=0}^{\infty} A_n^\alpha x^n = (1-x)^{-\alpha-1} \; (|x| < 1); \qquad ...(1.1.2)$$

and, by definition (Hardy 1949)

$$A_{-1}^\alpha = 0, \quad A_0^{-1} = 1, \quad A_n^{-\alpha} = 0 \quad (n \geq \alpha; \quad \alpha = 1, 2, \ldots) \qquad ...(1.1.3)$$

$$A_n^\alpha = \begin{cases} \binom{n+\alpha}{n} & (\alpha > -1), \\ (-1)^n \binom{-\alpha-1}{n} & (\alpha \leq -1); \end{cases} \qquad ...(1.1.4)$$

$$A_n^\alpha = \Gamma(n+\alpha+1) \{\Gamma(n+1)\Gamma(\alpha+1)\}^{-1} (\alpha \neq -1, -2, \ldots)$$
$$\sim n^\alpha \{\Gamma(\alpha+1)\}^{-1}. \qquad ...(1.1.5)$$

Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + \ldots + p_n \neq 0, \quad P_{-1} = p_{-1} = 0.$$

Then, the sequence-to-sequence transformation:

$$\tau_n = (P_n)^{-1} \sum_{\nu=0}^{n} p_{n-\nu} s_\nu,$$

defines the sequence $\{\tau_n\}$ of Nörlund means (Nörlund 1919, Woronoi 1932) of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$.

The series Σa_n is said to be absolutely summable (N, p_n), or simply summable $|N, p_n|$, if $\{\tau_n\} \in BV$, i.e., $\sum_n |\tau_n - \tau_{n-1}| \leqq K^*$ (Mears 1935).

In the special cases in which

$$p_n = A_n^{\alpha-1} \qquad (\alpha > -1); \qquad \ldots(1.1.6)$$

and

$$p_n = (n+1)^{-1}, \qquad P_n = 1 + \tfrac{1}{2} + \ldots + (n+1)^{-1} \sim \log n, \quad \text{as } n \to \infty, \qquad \ldots(1.1.7)$$

the Nörlund mean reduces respectively to the (C, α)-mean (Hardy 1949, § 5.13) and the harmonic mean (Hardy 1949, § 5.13; *see* also Riesz 1923) and the corresponding methods are then the same as $|C, \alpha|$-summability (Fekete 1911, Kogbetliantz 1925) and absolute harmonic summability.

The necessary and sufficient conditions for the regularity of the Nörlund mean, are:

$$\lim_{n\to\infty} \frac{p_n}{P_n} = 0, \qquad \ldots(1.1.8)$$

and

$$\sum_{\nu=0}^{n} |p_\nu| = O(|P_n|), \quad \text{as} \quad n \to \infty. \qquad \ldots(1.1.9)$$

If $\{p_n\}$ is real, non-negative and non-increasing sequence, then the conditions (1.1.8) and (1.1.9) are automatically satisfied. Thus, harmonic summability is regular.

We observe that, from (1.1.8),

$$\lim_{n\to\infty} \frac{P_{n-1}}{P_n} = \lim_{n\to\infty} \left(1 - \frac{p_n}{P_n}\right) = 1. \qquad \ldots(1.1.10)$$

For any sequence $\{u_n\}$, we write

$$\Delta^0 u_n = u_n, \Delta u_n = \Delta^1 u_n = u_n - u_{n+1}, \Delta^r \Delta^s u_n = \Delta^{r+s} u_n \qquad \ldots(1.1.11)$$

$$(r, s = 1, 2, \ldots),$$

* K denotes throughout an absolute constant, not necessarily the same at each occurrence.

and

$$\Delta^{\alpha} u_n = \sum_{\nu=0}^{\infty} A_{\nu}^{-\alpha-1} u_{\nu+n} \qquad \ldots(1.1.12)$$

provided the series on the right converges.

For $p \equiv p_n$, we also write

$$\left.\begin{array}{l} N^{\alpha}(p;\, n, \mu) \\ \bar{N}^{\alpha}(p;\, n, \mu) \end{array}\right\} = \sum_{\nu=0}^{n} A_{\nu}^{-\alpha-1} (P_{n+\mu}\, p_{n-\nu} - P_{n-\nu}\, p_{n+\mu}) \frac{\varepsilon_{\nu+\mu}}{\bar{\varepsilon}_{\nu+\mu}}, \qquad \ldots(1.1.13)$$

where $\bar{\varepsilon}_n = \Delta\varepsilon_n$, and

$$N^{\alpha}(\Delta p;\, n, \mu) = \sum_{\nu=0}^{n} A_{\nu}^{-\alpha-1} \Delta_{\nu}(P_{n+\mu}\, p_{n-\nu} - P_{n-\nu}\, p_{n+\mu})\, \varepsilon_{\nu+\mu}. \qquad \ldots(1.1.14)$$

1.2. Introduction

Chow (1954, Theorem 2) proved the following theorem for absolute Cesàro summability factors.

Theorem A—If $0 \leqq \beta \leqq \alpha, p \geq 0$, the necessary and sufficient conditions that $\Sigma\, a_n \varepsilon_n$ should be summable $|C, \beta|$, whenever $t_n^{\alpha} = 0(n^p)$, as $n \to \infty$, are:

(i) $\Sigma\, n^{\alpha-\beta-1+p}\, |\varepsilon_n| < \infty$, (ii) $\Sigma\, n^{\alpha+p}\, |\Delta^{\alpha}\, (n^{-1}\, \varepsilon_n)| < \infty$.

If $\beta > \alpha \geq 0$, the conditions are:

(i)′ $\Sigma\, n^{-1+p}\, |\varepsilon_n| < \infty$, (ii)′ $\Sigma\, n^{\alpha+p}\, |\Delta^{\alpha}\, (n^{-1}\, \varepsilon_n)| < \infty$.

Later, Bosanquet and Chow (1957) proved the following theorem by showing its equivalence to Theorem A cited above.

Theorem B—If $\alpha \geqq -1$, $\beta \geq 0, p \geq 0$, necessary and sufficient conditions for $\Sigma\, a_n \varepsilon_n$ to be summable $|C, \beta|$, whenever $s_n^{\alpha} = 0(n^p)$, as $n \to \infty$, are:

(Ia) $\Sigma\, n^{\alpha-\beta+p}\, |\varepsilon_n| < \infty$, (Ib) $\Sigma\, n^{-1+p}\, |\varepsilon_n| < \infty$,

(II) $\Sigma\, n^{\alpha+s}\, |\Delta^{\alpha+1}\, \varepsilon_n| < \infty$.

Since the demonstration of this theorem by Bosanquet and Chow was circuitous, involving the proof of a number of results which were apparently out of context, Pati and Ahmad (1960 a, b, c) in a series of three papers and Ahmad (1961) successfully generalized this theorem (Theorem B) in the case $\alpha \geq 0$, $\beta \geq 0$, by replacing the sequence $\{n^p\}$, $p \geq 0$, by the wider class of sequences $\{\lambda_n\}$, where $\{\lambda_n\}$ is a monotonic non-decreasing sequence, and by giving a direct and straightforward proof.

Theorem C—If $\alpha, \beta \geq 0$, necessary and sufficient conditions for $\Sigma\, a_n \varepsilon_n$ to be summable $| C, \beta |$ whenever $s_n^\alpha = O(\lambda_n)$, as $n \to \infty$, where $\{\lambda_n\}$ is a positive monotonic non-decreasing sequence, are:

(ia) $\Sigma\, n^{\alpha-\beta} \lambda_n | \varepsilon_n | < \infty$; (ib) $\Sigma\, n^{-1} \lambda_n | \varepsilon_n | < \infty$,

(ii) $\Sigma\, n^\alpha \lambda_n | \Delta^{\alpha+1} \varepsilon_n | < \infty$.

Recently, Ahmad and Khan (1974) have generalized Theorem A in the case $\alpha \geq 0$, for absolute Nörlund summability by replacing $| C, \beta |$ by $| N, p_n |$, where $\{p_n\}$ is a non-negative and non-increasing sequence, such that $p_0 > 0$ and $\{n^{-\alpha} P_n\}$ is bounded (this condition is omitted, when $\alpha = 0$ or $\alpha \geq 1$), and replacing the sequence $\{n^p\}$, $p \geq 0$, by a wider class of sequences $\{\lambda_n\}$. They proved:

Theorem D—Let $\alpha > 0$, and let $\{p_n\}$ be a non-negative, non-increasing sequence, such that $p_0 > 0$ and $\{n^{-\alpha} P_n\}$ is bounded*. Then, the necessary and sufficient conditions that the series $\Sigma\, a_n \varepsilon_n$ should be summable $| N, p_n |$ whenever $t_n^\alpha = O(\lambda_n)$, as $n \to \infty$, where $\{\lambda_n\}$ is a positive monotonic non-decreasing sequence, are:

(i) $\displaystyle\sum \frac{n^{\alpha-1}}{P_n} \lambda_n | \varepsilon_n | < \infty$,

and

(ii) $\Sigma\, n^\alpha \lambda_n | \Delta^\alpha (n^{-1} \varepsilon_n) | < \infty$,

or equivalently,

(ii)′ $\Sigma\, n^{\alpha-1} \lambda_n | \Delta^\alpha \varepsilon_n | < \infty$.

If $\alpha = 0$, the result holds without the hypothesis of boundedness of $\{n^{-\alpha} P_n\}$ and then the only necessary and sufficient condition is $\Sigma\, n^{-1} \lambda_n | \varepsilon_n | < \infty$.

Our aim here is to generalize Theorem C for absolute Nörlund summability in the same manner as Ahmad and Khan did to Theorem A.

2.1. We establish the following theorem.

Theorem—Let $\alpha > 0$, and let $\{p_n\}$ be a non-negative, non-increasing sequence such that $p_0 > 0$ and $\{n^{-\alpha} P_n\}$ is bounded. Then, the necessary and sufficient conditions that the series $\Sigma\, a_n \varepsilon_n$ should be summable $| N, p_n |$ whenever $s_n^\alpha = O(\lambda_n)$, as $n \to \infty$, where $\{\lambda_n\}$ is a positive monotonic non-decreasing sequence, are:

(i) $\displaystyle\sum \frac{n^\alpha}{P_n} \lambda_n |_n \varepsilon | < \infty$

and

(ii) $\Sigma\, n^\alpha \lambda_n | \Delta^{\alpha+1} \varepsilon_n | < \infty$.

If $\alpha = 0$, the result holds without the hypothesis of boundedness of $\{ n^{-\alpha} P_n\}$.

* The condition of boundedness of $\{n^{-\alpha} P_n\}$ is automatically satisfied when $\alpha \geqslant 1$.

Remark—Since p_n is non-negative and non-increasing such that $p_0 > 0$, the condition of boundedness of the sequence $\{n^{-\alpha} P_n\}$ is automatically satisfied, when $\alpha \geq 1$. Therefore, in our theorem, this should be treated as void for such α.

2.2. We need the following lemmas for the proof of our theorem.

Lemma 1 (Chow 1954, Lemma 6)—Let the sequences $\{U_n\}$ and $\{u_n\}$ be connected by the relation:

$$U_n = \sum_{\mu=1}^{n} a_{n\mu} u_\mu \qquad (n = 1, 2, \ldots). \qquad \ldots(2.2.1)$$

Then, a necessary condition for $\Sigma \mid U_n \mid < \infty$, whenever $u_n = O(1)$, is that

$$\sum_{\mu=1}^{\infty} \mid a_{\mu,\mu} \mid < \infty.$$

Lemma 2 (Chow 1954, Lemma 5)—Let the sequences $\{U_n\}$ and $\{u_n\}$ be connected by the relation (2.2.1). Then, a necessary condition for $\Sigma\ U_n < \infty$, whenever $u_n = O(1)$, is that:

$$\sum_{\mu=1}^{\infty} \Big| \sum_{n=1}^{\infty} a_{n\mu} \Big| < \infty.$$

Lemma 3—If $p_0 > 0$, and p_n is non-negative and non-increasing, then for $\nu \geq 1$, and $\mu \geq 1$,

$$(a) \quad \sum_{n=\nu}^{\infty} \frac{1}{P_n P_{n-1}} (P_n p_{n-\nu} - P_{n-\nu} p_n) \leq K;$$

$$(b) \quad \sum_{n=\nu}^{\infty} \frac{p_{n+\mu}}{P_{n+\mu} P_{n+\mu-1}} p_{n-\nu} \leq \frac{K}{P_{\nu+\mu}};$$

$$(c) \quad \sum_{n=\nu}^{\infty} \frac{\mid \Delta_n p_{n-\nu-1} \mid}{P_{n+\mu-1}} \leq \frac{K}{P_{\nu+\mu}}.$$

(*a*) is contained in Ahmad and Khan [1964, Lemma 3 (*c*)]. The proofs of (*b*) and (*c*) are similar to the proofs of Lemmas 1 and 3 of Ahmad (1966).

Lemma 4 (Ahmad and Khan 1964, Lemma 4)—If $p_0 > 0$ and p_n is non-negative and non-increasing, then, for $k \geq 0$ (k being finite), $\nu \geq 1$, and $\mu \geq 1$

$$\sum_{n=\nu}^{\infty} \frac{(P_{n+\mu} p_{n-\nu} - P_{n-\nu} p_{n+\mu})}{P_{n+\mu} P_{n+\mu-1}} \leq \frac{K}{P_{\nu+\mu}}.$$

Lemma 5 (Ahmad and Khan 1964, Lemma 11)—If $p_0 > 0$ and p_n is non-negative and non-increasing, then, for $\nu \geq 0$, and $\mu \geq 1$,

$$\sum_{n=\nu}^{\infty} \left(\frac{p_{n-\nu-1}}{P_{n+\mu-1}} - \frac{p_{n-\nu}}{P_{n+\mu}}\right) \leq \frac{p_\nu}{P_{\nu+\mu}}.$$

Lemma 6—Let $\alpha \geq 0$, and p_n be non-negative and non-increasing such that $p_0 > 0$. If $\varepsilon_n = O(1)$, then

$$\sum_{n=0}^{\infty} \frac{N^{\alpha+1}(p; n, \mu)}{P_{n+\mu}\, P_{n+\mu-1}} = \Delta^{\alpha+1} \varepsilon_\mu.$$

PROOF: By notation, from (1.1.13), we have

$$\sum_{n=0}^{\infty} \frac{N^{\alpha+1}(p; n, \mu)}{P_{n+\mu}\, P_{n+\mu-1}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{P_{n+\mu}\, P_{n+\mu-1}} \sum_{\nu=0}^{n} A_\nu^{-\alpha-2} (P_{n+\mu}\, p_{n-\nu} - P_{n-\nu}\, p_{n+\mu})\, \varepsilon_{\nu+\mu}$$

$$= \sum_{n=\mu}^{\infty} \frac{1}{P_n\, P_{n-1}} \sum_{\nu=\mu}^{n} A_{\nu-\mu}^{-\alpha-2} (P_n\, p_{n-\nu} - P_{n-\nu}\, p_n)\, \varepsilon_\nu$$

$$= \sum_{\nu=\mu}^{\infty} A_{\nu-\mu}^{-\alpha-2}\, \varepsilon_\nu \sum_{n=\nu}^{\infty} \left(\frac{P_n\, p_{n-\nu} - p_n\, P_{n-\nu}}{P_n\, P_{n-1}}\right)$$

[interchange of the order of summation being legitimate here due to absolut convergence of the double series by virtue of the hypothesis and Lemma 3 (*a*)]

$$= \sum_{\nu=\mu}^{\infty} A_{\nu-\mu}^{-\alpha-2}\, \varepsilon_\nu \sum_{n=\nu}^{\infty} \left(\frac{P_{n-\nu}}{P_n} - \frac{P_{n-\nu-1}}{P_{n-1}}\right)$$

$$= \sum_{\nu=\mu}^{\infty} A_{\nu-\mu}^{-\alpha-2}\, \varepsilon_\nu \left\{\frac{P_0}{P_\nu} - \frac{P_{-1}}{P_{\nu-1}} + \frac{P_1}{P_{\nu+1}} - \frac{P_0}{P_\nu} + \ldots + \right.$$

$$\left. + \lim_{m\to\infty} \frac{P_{m-\nu}}{P_m} - \lim_{m\to\infty} \frac{P_{m-\nu-1}}{P_{m-1}}\right\}$$

$$= \sum_{\nu=\mu}^{\infty} A_{\nu-\mu}^{-\alpha-2}\, \varepsilon_\nu \left\{\lim_{m\to\infty} \frac{P_{m-\nu}}{P_m}\right\}$$

$$= \sum_{\nu=\mu}^{\infty} A_{\nu-\mu}^{-\alpha-2}\, \varepsilon_\nu = \Delta^{\alpha+1} \varepsilon_\mu,$$

by regularity of the Nörlund mean and (1.1.10), and since, by hypothesis, the series

$$\sum_{\nu=\mu}^{\infty} A_{\nu-\mu}^{-\alpha-2}\ \varepsilon_\nu$$

is absolutely convergent.

This completes the proof of Lemma 6.

2.3. Proof of the Theorem

Let $\tau_n{}^*$ be the nth Nörlund mean of the series $\sum_{\nu=0}^{\infty} \varepsilon_\nu a_\nu$. Then, by definition

$$\tau_n{}^* = \frac{1}{P_n} \sum_{\nu=0}^{n} P_{n-\nu}\, \varepsilon_\nu\, a_\nu,$$

and hence, by Abel's transformation,

$$\tau_n^* - \tau_{n-1}^* = \frac{1}{P_n P_{n-1}} \sum_{\nu=1}^{n} (P_n p_{n-\nu} - P_{n-\nu} p_n)\, \varepsilon_\nu\, a_\nu$$

$$= \frac{1}{P_n P_{n-1}} \sum_{\nu=1}^{n} \Delta_\nu \left[(P_n p_{n-\nu} - P_{n-\nu} p_n)\ \varepsilon_\nu\right] s_\nu$$

$$+ \frac{s_n}{P_n P_{n-1}} \left[(P_n p_{n-\nu-1} - P_{n-\nu-1} p_n)\, \varepsilon_{\nu+1}\right]_{\nu=n}$$

$$= \frac{1}{P_n P_{n-1}} \sum_{\nu=1}^{n} \Delta_\nu \left[(P_n p_{n-\nu} - P_{n-\nu} p_n)\, \varepsilon_\nu\right]$$

$$\times \sum_{\mu=1}^{\nu} A_{\nu-\mu}^{-\alpha-1} A_\mu^\alpha s_\mu^\alpha$$

$$= \frac{1}{P_n P_{n-1}} \sum_{\mu=1}^{n} A_\mu^\alpha s_\mu^\alpha \sum_{\nu=\mu}^{n} A_{\nu-\mu}^{-\alpha-1}$$

$$\times \Delta_\nu \left[(P_n p_{n-\nu} - P_{n-\nu} p_n)\ \varepsilon_\nu\right]$$

$$= \frac{1}{P_n P_{n-1}} \sum_{\mu=1}^{n} A_\mu^\alpha s_\mu^\alpha N^{\alpha+1}(p;\ n-\mu, \mu), \qquad \text{...(2.3.1)}$$

since, by notation (1.1.13) and by Abel's transformation,

$$N^{\alpha+1}(p; n-\mu, \mu)$$
$$= \sum_{\nu=0}^{n-\mu} A_{\nu}^{-\alpha-2}(P_n p_{n-\nu-\mu} - P_{n-\nu-\mu} p_n)\, \varepsilon_{\nu+\mu}$$
$$= \sum_{\nu=0}^{n-\mu} A_{\nu}^{-\alpha-1} \triangle_{\nu} [(P_n p_{n-\nu-\mu} - P_{n-\nu-\mu} p_n)\, \varepsilon_{\nu+\mu}]$$
$$+ A_{n}^{-\alpha-1} [(P_n p_{n-\nu-\mu-1} - P_{n-\nu-\mu-1} p_n)\, \varepsilon_{\nu+\mu+1}]_{\nu=n-\mu}$$
$$= \sum_{\nu=0}^{n-\mu} A_{\nu}^{-\alpha-1} \Delta_{\nu} [(P_n p_{n-\nu-\mu} - P_{n-\nu-\mu} p_n)\, \varepsilon_{\nu+\mu}]$$
$$= \sum_{\nu=\mu}^{n} A_{\nu-\mu}^{-\alpha-1} \Delta_{\nu} [(P_n p_{n-\nu} - P_{n-\nu} p_n)\, \varepsilon_{\nu}]\,.$$

Thus, we have

$$U_n = \tau_n^* - \tau_{n-1}^* = \sum_{\mu=1}^{n} \frac{A_{\mu}^{\alpha} \lambda_{\mu} N^{\alpha+1}(p; n-\mu, \mu)}{P_n P_{n-1}} \left(\frac{s_{\mu}^{\alpha}}{\lambda_{\mu}}\right)$$
$$= \sum_{\mu=1}^{n} a_{n\mu} u_{\mu} \qquad \ldots(2.3.2)$$

where

$$a_{n\mu} = \frac{A_{\mu}^{\alpha} \lambda_{\mu} N^{\alpha+1}(p; n-\mu, \mu)}{P_n P_{n-1}};$$
$$u_{\mu} = \left(\frac{s_{\mu}^{\alpha}}{\lambda_{\mu}}\right).$$

Necessity

The condition (i) is necessary. For, from (2.3.2) in order that $\Sigma \mid \tau_n^* - \tau_{n-1}^* \mid = \Sigma \mid U_n \mid < \infty$, whenever $u_n = O(1)$, by Lemma 1, it is necessary that

$$\sum_{\mu=1}^{\infty} \mid a_{\mu\mu} \mid < \infty,$$

but

$$a_{\mu\mu} = \frac{A_{\mu}^{\alpha} \lambda_{\mu}}{P_{\mu} P_{\mu-1}} N^{\alpha+1}(p; 0, \mu)$$
$$= \frac{A_{\mu}^{\alpha} \lambda_{\mu}}{P_{\mu} P_{\mu-1}} p_0\, \varepsilon_{\mu} P_{\mu-1}$$
$$= p_0 \frac{A_{\mu}^{\alpha} \lambda_{\mu} \varepsilon_{\mu}}{P_{\mu}},$$

so that

$$\sum_{\mu=1}^{\infty} a_{\mu\mu} = \sum_{\mu=1}^{\infty} p_0 \frac{A_{\mu}^{\alpha}}{P_{\mu}} \lambda_{\mu} \mid \varepsilon_{\mu} \mid < \infty,$$

i.e.,

$$\sum_{\mu=1}^{\infty} \frac{\mu^{\alpha}}{P_{\mu}} | \varepsilon_{\mu} | < \infty$$

since $p_0 > 0$.

The condition (ii) is necessary. First considering the case $\alpha=0$, we see that

$$\begin{aligned} a_{n\mu} &= \frac{\lambda_{\mu}}{P_n P_{n-1}} N^1(p; n-\mu, \mu) \\ &= \frac{\lambda_{\mu}}{P_n P_{n-1}} \Delta_{\mu} [(P_n p_{n-\mu} - P_{n-\mu} p_n) \varepsilon_{\mu}] \\ &= \frac{\lambda_{\mu}}{P_n P_{n-1}} [\Delta \varepsilon_{\mu} (P_n p_{n-\mu} - P_{n-\mu} p_n) \\ &\qquad + \varepsilon_{\mu+1} \Delta_{\mu} (P_n p_{n-\mu} - P_{n-\mu} p_n)] \end{aligned}$$

so that,

$$\begin{aligned} \left| \sum_{n=\mu}^{\infty} a_{n\mu} \right| &\geq \lambda_{\mu} | \Delta \varepsilon_{\mu} | \sum_{n=\mu}^{\infty} (P_n p_{n-\mu} - P_{n-\mu} p_n)/P_n P_{n-1} \\ &\quad - \lambda_{\mu} | \varepsilon_{\mu+1} | \left| \sum_{n=\mu}^{\infty} \Delta_{\mu} (P_n p_{n-\mu} - P_{n-\mu} p_n)/P_n P_{n-1} \right| \qquad \ldots(2.3.3) \end{aligned}$$

Therefore, in order that $\Sigma\, U_n < \infty$, whenever $u_n = O(1)$, by Lemma 2, it is necessary that

$$\sum_{\mu=1}^{\infty} \sum_{n=\mu}^{\infty} | a_{n\mu} | < \infty,$$

i.e.,

$$\sum_{\mu=1}^{\infty} \left| \sum_{\mu=1}^{\infty} a_{n\mu} \right| \leq K,$$

so that, by (2.3.3), we have

$$\begin{aligned} &\sum_{\mu=1}^{\infty} \lambda_{\mu} | \Delta\varepsilon_{\mu} | \sum_{n=\mu}^{\infty} \frac{(P_n p_{n-\mu} - P_{n-\mu} p_n)}{P_n P_{n-1}} \\ &\leq \sum_{\mu=1}^{\infty} \left| \sum_{n=\mu}^{\infty} a_{n\mu} \right| + \sum_{\mu=1}^{\infty} \lambda_{\mu} | \varepsilon_{\mu+1} | \\ &\quad \times \left| \sum_{n=\mu}^{\infty} \frac{\Delta_{\mu} (P_n p_{n-\mu} - P_{n-\mu} p_n)}{P_n P_{n-1}} \right| \end{aligned}$$

or

$$\sum_{\mu=1}^{\infty} \lambda_\mu \,|\, \Delta \varepsilon_\mu \,|\, \sum_{n=\mu}^{\infty} \frac{(P_n p_{n-\mu} - P_{n-\mu} p_n)}{P_n P_{n-1}}$$

$$\leqq K + \sum_{\mu=1}^{\infty} \lambda_\mu \,|\, \varepsilon_{\mu+1} \,|\, \left(\sum_{n=\mu}^{\infty} \frac{|\Delta_\mu p_{n-\mu}|}{P_{n-1}} + \sum_{n=\mu}^{\infty} \frac{p_n p_{n-\nu}}{P_n P_{n-1}} \right)$$

$$\leqq K + K \sum_{\mu=1}^{\infty} \frac{\lambda_\mu \,|\, \varepsilon_{\mu+1} \,|}{P_{\mu+1}} \quad \text{[by Lemmas 3 } (b) \text{ and 3 } (c)]$$

$$\leqq K,$$

by condition (i).

But, since as in the proof of Lemma 6,

$$\sum_{n=\mu}^{\infty} \frac{(P_n p_{n-\mu} - P_{n-\mu} p_n)}{P_n P_{n-1}} = \lim_{m \to \infty} \frac{P_{m-\mu}}{P_m} = 1,$$

for a fixed μ, by regularity of Nörlund method (1.1.10), we get

$$\sum_{\mu=1}^{\infty} \lambda_\mu \,|\, \Delta \varepsilon_\mu \,| < \infty.$$

Next, we consider the case $\alpha > 0$. Since, under the hypothesis of the theorem, condition (i) implies $\varepsilon_n = O(1)$, in order that $\Sigma\, U_n < \infty$, whenever $u_n = O(1)$, by Lemma 1, it is necessary that

$$\sum_{\mu=1}^{\infty} \Big| \sum_{n=1}^{\infty} a_{n\mu} \Big| < \infty,$$

i.e.,

$$\sum_{\mu=1}^{\infty} \mu^\alpha \lambda_\mu \left| \sum_{n=\mu}^{\infty} \frac{N^{\alpha+1}(p; n-\mu, \mu)}{P_n P_{n-1}} \right| = \sum_{\mu=1}^{\infty} \mu^\alpha \lambda_\mu \,|\, \Delta^{\alpha+1} \varepsilon_\mu | < \infty,$$

by Lemma 5.

This completes the proof of the necessity part.

Sufficiency

From (2.3.1), we have

$$\tau_n^* - \tau_{n-1}^* = \frac{1}{P_n P_{n-1}} \sum_{\mu=1}^{n} A_\mu^\alpha s_\mu^\alpha \sum_{\nu=\mu}^{n} A_{\nu-\mu}^{-\alpha-1} \Delta_\nu [(P_n p_{n-\nu} - P_{n-\nu} p_n) \varepsilon_\nu]$$

$$= \frac{1}{P_n P_{n-1}} \sum_{\mu=1}^{n} A_\mu^\alpha s_\mu^\alpha \sum_{\nu=\mu}^{n} A_{\nu-\mu}^{-\alpha-1} (P_n p_{n-\nu} - P_{n-\nu} p_n) \Delta\varepsilon_\nu$$

$$+ \frac{1}{P_n P_{n-1}} \sum_{\mu=1}^{n} A_\mu^\alpha s_\mu^\alpha \sum_{\nu=\mu}^{n} A_{\nu-\mu}^{-\alpha-1} \Delta_\nu (P_n p_{n-\nu} - P_{n-\nu} p_n) \varepsilon_{\nu+1}$$

$$= \frac{1}{P_n P_{n-1}} \sum_{\mu=1}^{n} A_\mu^\alpha s_\mu^\alpha \bar{N}^\alpha (p; n-\mu, \mu)$$

$$+ \frac{1}{P_n P_{n-1}} \sum_{\mu=1}^{n} A_\mu^\alpha s_\mu^\alpha N^\alpha (\Delta p; n-\mu, \mu+1).$$

Hence,

$$\sum_{n=1}^{\infty} | \tau_n^* - \tau_{n-1}^* | \leqq \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{\mu=1}^{n} A_\mu^\alpha | s_\mu^\alpha | \; | \bar{N}^\alpha (p; n-\mu, \mu) |$$

$$+ \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{\mu=1}^{n} A_\mu^\alpha | s_\mu^\alpha | \, | N^\alpha (\Delta p; n-\mu, \mu+1 |.$$

Therefore, in order to prove the theorem, it is sufficient to prove that

$$\sum\nolimits_1 = \sum_{\mu=1}^{\infty} A_\mu^\alpha \lambda_\mu \sum_{n=0}^{\infty} \frac{| \bar{N}^\alpha (p; n, \mu) |}{P_{n+\mu} P_{n+\mu-1}} < \infty, \qquad \ldots(2.3.4)$$

$$\sum\nolimits_2 = \sum_{\mu=1}^{\infty} A_\mu^\alpha \lambda_\mu \sum_{n=0}^{\infty} \frac{| N^\alpha (\Delta p; n, \mu) |}{P_{n+\mu} P_{n+\mu-1}} < \infty. \qquad \ldots(2.3.5)$$

Proof of (2.3.4): *Case I*—When $\alpha = 0$. In this case, we have

$$\sum\nolimits_1 = \sum_{\mu=1}^{\infty} \lambda_\mu \sum_{n=0}^{\infty} \frac{| \bar{N}^0 (p; n, \mu) |}{P_{n+\mu} P_{n+\mu-1}}$$

$$= \sum_{\mu=1}^{\infty} \lambda_\mu \sum_{n=0}^{\infty} \frac{(P_{n+\mu} p_n - P_n p_{n+\mu})}{P_{n+\mu} P_{n+\mu-1}} \varepsilon_\mu$$

$$= \sum_{\mu=1}^{\infty} \lambda_\mu \,|\, \Delta \varepsilon_\mu \,|\, \sum_{n=\mu}^{\infty} \frac{(P_n p_{n-\mu} - P_{n-\mu} p_n)}{P_n P_{n-1}}$$

$$\leqq K \sum_{\mu=1}^{\infty} \lambda_\mu \,|\, \Delta \varepsilon_\mu \,| < K,$$

by hypothesis and Lemma 3 (a).

Case II—When $\alpha > 0$—We write

$$\Sigma_1 = \sum_{\mu=1}^{\infty} A_\mu^\alpha \, \lambda_\mu \, [\sigma_1(\mu) + (\sigma_2(\mu)],$$

say, where

$$\sigma_1(\mu) = \sum_{n=0}^{k} \frac{|\, \bar{N}^\alpha (p; n, \mu) \,|}{P_{n+\mu} P_{n+\mu-1}}$$

and

$$\sigma_2(\mu) = \sum_{n=k+1}^{\infty} \frac{|\, \bar{N}^\alpha (p; n, \mu) \,|}{P_{n+\mu} P_{n+\mu-1}}$$

Therefore, in order that $\Sigma_1 \leq K$, it is sufficient to show that

$$\Sigma_{1,r} = \sum_{\mu=1}^{\infty} A_\mu^\alpha \, \lambda_\mu \, \sigma_r(\mu) \leq K, \qquad (r = 1, 2). \qquad \ldots(2.3.6):(r)$$

PROOF OF (2.3.6) : (1)—Since

$$\sigma_1(\mu) = \sum_{n=0}^{k} \left| \sum_{\nu=0}^{n} A_\nu^{-\alpha-1} \frac{(P_{n+\mu} p_{n-\nu} - P_{n-\nu} p_{n+\mu})}{P_{n+\mu} P_{n+\mu-1}} \, \bar{\epsilon}_{\nu+\mu} \right|$$

$$\leqq \sum_{\nu=0}^{k} |\, \bar{\epsilon}_{\nu+\mu} \,|\, |\, A_\nu^{-\alpha-1} \,| \sum_{n=\nu}^{k} \frac{(P_{n+\mu} p_{n-\nu} - P_{n-\nu} p_{n-\mu})}{P_{n+\mu} P_{n+\mu-1}}$$

$$\leqq K \sum_{\nu=0}^{k} |\, \Delta \varepsilon_{\nu+\mu} \,|\, \frac{1}{P_{\nu+\mu}} \text{ (by Lemma 4)},$$

we have

$$\sum\nolimits_{1,1} \leqq K \sum_{\mu=1}^{\infty} A_{\mu}^{\alpha} \lambda_{\mu} \sum_{\nu=0}^{k} |\Delta \varepsilon_{\nu+\mu}| \frac{1}{P_{\nu+\mu}}$$

$$\leqq K \sum_{\nu=0}^{k} \sum_{\mu=1}^{\infty} \frac{(\nu+\mu)^{\alpha}}{P_{\nu+\mu}} |\Delta \varepsilon_{\nu+\mu}| \leq K,$$

by hypothesis and condition (i).

PROOF OF (2.3.6) : (2)—Since

$$\bar{N}^{\alpha}(p, n, \mu)$$

$$= \sum_{\nu=0}^{n} A_{\nu}^{-\alpha-1} (P_{n+\mu} p_{n-\nu} - P_{n-\nu} p_{n+\mu}) \bar{\varepsilon}_{\nu+\mu}$$

$$= \sum_{\nu=0}^{n} \Delta_{\nu} (P_{n+\mu} p_{n-\nu} - P_{n-\nu} p_{n+\mu}) \sum_{j=0}^{\nu} A_{j}^{-\alpha-1} \bar{\varepsilon}_{j+\mu}$$

$$+ [(P_{n+\mu} p_{n-\nu-1} - P_{n-\nu-1} p_{n+\mu})]_{\nu=n} \sum_{j=0}^{n} A_{j}^{-\alpha-1} \bar{\varepsilon}_{j+\mu}$$

$$= \sum_{\nu=0}^{n} \Delta_{\nu} (P_{n+\mu} p_{n-\nu} - P_{n-\nu} p_{n+\mu}) \sum_{j=0}^{\nu} A_{j}^{-\alpha-1} \bar{\varepsilon}_{j+\mu}$$

$$= \bar{N}_{1}^{\alpha}(p; n, \mu) + \bar{N}_{2}^{\alpha}(p; n, \mu) + \bar{N}_{3}^{\alpha}(p; n, \mu), \qquad \ldots(2.3.7)$$

say, where

$$\bar{N}_{1}^{\alpha}(p, n, \mu) = (P_{n+\mu} p_{n} - p_{n+\mu} P_{n}) \Delta^{\alpha} \bar{\varepsilon}_{\mu}, \qquad \ldots(2.3.8)$$

$$\bar{N}_{2}^{\alpha}(p, n, \mu) = -(P_{n+\mu} p_{n} - P_{n} p_{n+\mu}) \sum_{j=n+1}^{\infty} A_{j}^{-\alpha-1} \bar{\varepsilon}_{j+\mu}\ ; \qquad \ldots(2.3.9)$$

and

$$\bar{N}_{3}^{\alpha}(p, n, \mu) = - \sum_{\nu=0}^{n} \Delta_{\nu} (P_{n+\mu} p_{n-\nu} - P_{n-\nu} p_{n+\mu}) \sum_{j=\nu+1}^{n} A_{j}^{-\alpha-1} \bar{\varepsilon}_{j+\mu} \qquad \ldots(2.3.10)$$

in order to prove $\Sigma_{1,2} \leqq K$, it is sufficient to show that,

$$\sum\nolimits_{1,2,s} = \sum_{\mu=1}^{\infty} A_{\mu}^{\alpha} \lambda_{\mu} \sum_{n=k+1}^{\infty} \frac{|\bar{N}_{s}^{\alpha}(p; n, \mu)|}{P_{n+\mu} P_{n+\mu-1}} \leqq K, \quad (s = 1, 2, 3) \qquad \ldots(2.3.11)$$

PROOF OF (2.3.11)—We can easily see that for $n > k$, $\bar{N}_s^\alpha(p; n, \mu) \equiv 0$, for $s = 2, 3$, when α is an integer.

By (2.3.8) we have

$$\sum\nolimits_{1,2,1} < \sum_{\mu=1}^{\infty} A_\mu^\alpha \lambda_\mu \sum_{n=0}^{\infty} \frac{(P_{n+\mu} p_n - p_{n+\mu} P_n)}{P_{n+\mu} P_{n+\mu}} | \Delta^\alpha \bar{\varepsilon}_\mu |$$

$$\leqq K \sum_{\mu=1}^{\infty} \mu^\alpha \lambda_\mu | \Delta^\alpha \bar{\varepsilon}_\mu | \sum_{n=\mu}^{\infty} \frac{(P_n p_{n-\mu} - P_{n-\mu} p_n)}{P_n P_{n-1}}$$

$$\leqq K \sum_{\mu=1}^{\infty} \mu^\alpha \lambda_\mu | \Delta^{\alpha+1} \varepsilon_\mu | \text{ [by Lemma 3 (a)]}$$

$$\leqq K,$$

by hypothesis and condition (ii).

Next, by (2.3.9), we have

$$\sum\nolimits_{1,2,2} < \sum_{\mu=1}^{\infty} A_\mu^\alpha \lambda_\mu \sum_{n=0}^{\infty} \frac{(P_{n+\mu} p_n - P_n p_{n+\mu})}{P_{n+\mu} P_{n+\mu-1}}$$

$$\times \sum_{j=n+1}^{\infty} | A_j^{-\alpha-1} | | \bar{\varepsilon}_{j+\mu} |$$

$$= \sum_{\mu=1}^{\infty} A_\mu^\alpha \lambda_\mu \sum_{j=1}^{\infty} | A_j^{-\alpha-1} | | \bar{\varepsilon}_{j+\mu} |$$

$$\times \sum_{n=\mu}^{\mu+j-1} \frac{(P_n p_{n-\mu} - P_{n-\mu} p_n)}{P_n P_{n-1}}$$

$$= \sum_{\mu=1}^{\infty} A_\mu^\alpha \lambda_\mu \sum_{j=1}^{\infty} | A_j^{-\alpha-1} | | \bar{\varepsilon}_{j+\mu} |$$

$$\times \sum_{n=\mu}^{\mu+j-1} \left(\frac{P_{n-\mu}}{P_n} - \frac{P_{n-\mu-1}}{P_{n-1}} \right)$$

$$= \sum_{\mu=1}^{\infty} A_\mu^\alpha \lambda_\mu \sum_{j=1}^{\infty} | A_j^{-\alpha-1} | | \bar{\varepsilon}_{j+\mu} | \frac{P_{j-1}}{P_{j+\mu-1}}$$

$$< K \sum_{\mu=1}^{\infty} \mu^\alpha \lambda_\mu \sum_{j=1}^{\infty} | A_j^{-\alpha-1} | \frac{| \bar{\varepsilon}_{j+\mu} |}{P_{j+\mu-1}} \left(\frac{P_j}{j^\alpha} \right) j^\alpha$$

$$\leqq K \sum_{\mu=1}^{\infty} \mu^{a} \lambda_{\mu} \sum_{j=1}^{\infty} | A_{j-1}^{-1} \quad | \frac{|\bar{\varepsilon}_{j+\mu}|}{P_{j+\mu-1}}$$

$$= K \sum_{\mu=1}^{\infty} \frac{\mu^{a}}{P_{\mu}} | \Delta \varepsilon_{\mu+1} | \leq K,$$

by hypothesis and condition (i).

Lastly, by (2.3.10), we have

$$\sum\nolimits_{1,2,3} < \sum_{\mu=1}^{\infty} A_{\mu}^{a} \lambda_{\mu} \sum_{n=0}^{\infty} \frac{1}{P_{n+\mu} P_{n+\mu-1}}$$

$$\times \sum_{\nu=0}^{n} | \Delta_{\nu} (P_{n+\mu} p_{n-\nu} - P_{n-\nu} p_{n+\mu}) | \sum_{j=\nu+1}^{n} | A_{j}^{-a-1} | | \bar{\varepsilon}_{j+\mu} |$$

$$= \sum_{\mu=1}^{\infty} A_{\mu}^{a} \lambda_{\mu} \sum_{\nu=0}^{\infty} \sum_{j=\nu+1}^{\infty} | A_{j}^{-a-1} | | \bar{\varepsilon}_{j+\mu} | \sum_{n=\nu}^{\infty} \frac{| \Delta_{\nu} (P_{n+\mu} p_{n-\nu} - P_{n-\nu} p_{n+\mu}) |}{P_{n+\mu} P_{n+\mu-1}}$$

(since the series $\sum_{j=\nu+1}^{\infty} | A_{j}^{-a-1} | | \bar{\varepsilon}_{j+\mu} |$ is convergent).

$$= \sum_{\mu=1}^{\infty} A_{\mu}^{a} \lambda_{\mu} \sum_{j=1}^{\infty} | A_{j}^{-a-1} | | \bar{\varepsilon}_{j+\mu} |$$

$$\times \sum_{\nu=0}^{j-1} \sum_{n=\nu}^{\infty} \left(\frac{p_{n-\nu-1}}{P_{n+\mu-1}} - \frac{p_{n-\nu}}{P_{n+\mu}} \right)$$

$$\leqq \sum_{\mu=1}^{\infty} A_{\mu}^{a} \lambda_{\mu} \sum_{j=1}^{\infty} | A_{j}^{-a-1} | | \bar{\varepsilon}_{j+\mu} | \sum_{\nu=0}^{j-1} \frac{p_{\nu}}{P_{\nu+\mu}}$$

(by Lemma 5)

$$\leqq K \sum_{\mu=1}^{\infty} \frac{\mu^{a} \lambda_{\mu}}{P_{\mu}} \sum_{j=1}^{\infty} | A_{j}^{-a-1} | | \bar{\varepsilon}_{j+\mu} | P_{j-1}$$

$$< K \sum_{\mu=1}^{\infty} \frac{\mu^{a} \lambda_{\mu}}{P_{\mu}} \sum_{j=1}^{\infty} | A_{j}^{-a-1} | | \bar{\varepsilon}_{j+\mu} | \left(\frac{P_{j}}{j^{a}} \right) j^{a}$$

$$\leqq K \sum_{\mu=1}^{\infty} \frac{\mu^{\alpha} \lambda_{\mu}}{P_{\mu}} \sum_{j=1}^{\infty} | A_{j-1}^{-1} | | \bar{\varepsilon}_{j+\mu} |$$

$$= K \sum_{\mu=1}^{\infty} \frac{\mu^{\alpha}}{P_{\mu}} \lambda_{\mu} | \Delta \varepsilon_{j+1} | \leq K,$$

by hypothesis and condition (i).

This completes the proof of (2.3.11).

PROOF OF (2.3.5)—For $\alpha \geq 0$ we have

$$\sum\nolimits_2 \leqq \sum_{\mu=1}^{\infty} A_{\mu}^{\alpha} \lambda_{\mu} \sum_{n=0}^{\infty} \frac{1}{P_{n+\mu} P_{n+\mu-1}} \sum_{\nu=0}^{\infty} | A_{\nu}^{-\alpha-1} |$$

$$\times | \Delta_{\nu} (P_{n+\mu} p_{n-\nu} - P_{n-\nu} p_{n+\mu}) | | \varepsilon_{\nu+\mu+1} |$$

$$= \sum_{\mu=1}^{\infty} A_{\mu}^{\alpha} \lambda_{\mu} \sum_{\nu=0}^{\infty} | A_{\nu}^{-\alpha-1} | |\varepsilon_{\nu+\mu+1} |$$

$$\times \left(\sum_{n=0}^{\infty} \frac{| \Delta_n p_{n-\nu-1} |}{P_{n+\mu-1}} + \sum_{n=\nu}^{\infty} \frac{p_{n+\mu}}{P_{n+\mu} P_{n+\mu-1}} p_{n-\nu} \right)$$

$$\leqq K \sum_{\mu=1}^{\infty} A_{\mu}^{\alpha} \lambda_{\mu} \sum_{\nu=0}^{\infty} | A_{\nu}^{-\alpha-1} | | \varepsilon_{\nu+\mu+1} | \frac{1}{P_{\nu+\mu}}$$

[by Lemmas 3 (*b*) and 3 (*c*)]

$$= K \sum_{\mu=1}^{\infty} A_{\mu}^{\alpha} \lambda_{\mu} \sum_{\nu=\mu}^{\infty} | A_{\nu-\mu}^{-\alpha-1} | \frac{| \varepsilon_{\nu+1} |}{P_{\nu+1}}$$

$$= K \sum_{\nu=1}^{\infty} \frac{1}{P_{\nu+1}} | \varepsilon_{\nu+1} | \sum_{\mu=1}^{\nu} | A_{\nu-\mu}^{-\alpha-1} | A_{\mu}^{\alpha} \lambda_{\mu}$$

$$\leqq K \sum_{\nu=1}^{\infty} \frac{\nu^{\alpha}}{P_{\nu}} | \varepsilon_{\nu+1} | \sum_{\mu=1}^{\nu} | A_{\nu-\mu}^{-\alpha-1} |$$

$$\leqq K \sum_{\nu=1}^{\infty} \frac{\nu^{\alpha}}{P_{\nu}} \lambda_{\mu} | \varepsilon_{\nu+1} | \leq K,$$

by hypothesis and condition (i).

This completes the proof of (2.3.5) and hence terminates the proof of the theorem.

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