GRAPHOIDAL COVERS AND GRAPHOIDAL COVERING NUMBER OF A GRAPH

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Given a graph \( G = (V, E) \), a graphoid on \( G \) is defined here to be a collection \( \Psi \) of (not necessarily open) paths in \( G \) such that (a) every path in \( \Psi \) has at least two points, and (b) every point of \( G \) is an internal point of at most one path in \( \Psi \). If, further, (c) every line of \( G \) is in some path in \( \Psi \), then we call \( \Psi \) a graphoidal cover of \( G \). The graphoidal covering number of \( G \), denoted \( \gamma (G) \), is then defined to be the minimum cardinality of a graphoidal cover of \( G \). The basic preoccupation of this paper is an attempt to find suitable bounds for \( \gamma \). In the sequel, we obtain a characterization of graphs \( G \) which admit a labeling \( f \) of the points of \( G \) by the numbers 1, 2, \ldots, \( n \), \( n = 1 \; V(G) \; 1 \), so that the set of maximal directed paths in the low-to-high orientation of \( G \) with respect to \( f \) is a graphoidal cover of \( G \). We then discuss the problem of representing a graph as an intersection graph of a graphoidal cover of a graph. Several other key problems are also proposed or indicated.

INTRODUCTION

For all terminology and notation in graph theory, digraph theory and hypergraph theory we follow references. Harary\(^7\), Harary et al.\(^2\) and Berge\(^3\) respectively.

If \( P = (u_0, u_1, u_2, \ldots, u_t) \) is a path, not necessarily open, in a graph \( G = (V, E) \) then \( u_0 \) and \( u_t \) are called terminals of \( P \) and \( u_1, u_2, \ldots, u_{t-1} \) are called transits or internal points of \( P \). Furthermore, \( P' = (u_t, u_{t-1}, \ldots, u_1, u_0) \), the reverse of \( P \), is regarded equivalent to \( P \).

Given a graph \( G = (V, E) \), by a graphoid on \( G \) we mean a set \( \Psi \) of (not necessarily open) paths in \( G \) satisfying the following two conditions:

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C1. Every path in $\Psi$ has at least two points.

C2. Every point of $G$ is a transit of at most one path in $\Psi$.

Further, $\Psi$ is called a graphoidal cover of $G$ if, in addition to C1 and C2, it satisfies

C3. Every line of $G$ is in some path in $\Psi$.

In any graph $G = (V, E)$ without isolated points (we will assume this throughout this paper), the collection $E(G)$ of its lines itself is a graphoidal cover of $G$ as it satisfies C1, C2 (trivially) and C3. Thus, the set $\mathcal{C}(G)$ of all graphoidal covers of $G$ is nonempty in general. Also, it is a finite set whenever $G$ is so (this too is assumed throughout). Hence

$$\gamma(G) = \min_{\Psi \in \mathcal{C}(G)} |\Psi|$$

is well defined for any graph $G$ and is referred to as the graphoidal covering number of $G$ where $|\Psi|$ denotes the cardinality of the set $\Psi$. A graphoidal cover $\Psi$ of $G$ is said to be minimum if $|\Psi| = \gamma(G)$.

RESULTS

For any graph $G$, since $E(G) \in \mathcal{C}(G)$ we have $\gamma(G) \leq |E(G)|$. Next, let $t(P)$ denote the number of transits in the path $P$. Then, for any graphoidal cover $\{P_1, P_2, \ldots, P_k\}$ of $G$ we have

$$|E(G)| \leq \sum_{i=1}^{k} |E(P_i)| = k + \sum_{i=1}^{k} t(P_i)$$

and since every point of $G$ is a transit of at most one of the $P_i$'s we have

$$\sum_{i=1}^{k} t(P_i) \leq |V(G)|$$

invoking which in the previous inequality yields $k \geq |E(G)| - |V(G)|$. Thus, we have the bounds for $\gamma(G)$:

$$|E(G)| - |V(G)| \leq \gamma(G) \leq |E(G)|.$$

...(2)

**Theorem 1**—For the complete graph $K_n$ of order $n \geq 2$, we have

(a) $\gamma(K_2) = \gamma(K_3) = 1$,

(b) $\gamma(K_n) = n(n - 3)/2$ for $n \geq 4$.

**Proof**: Since $K_2$ is a path (of length one) and $K_3$ is a cycle (of length three), (a) follows easily. Hence, let $n \geq 4$ and $V(K_n) = \{1, 2, \ldots, n\} = n$. The graphoidal cover $\Psi = \{P = (1, 2, \ldots, n), Q = (3, 1, n, 2)\} \cup (E(K_n) - E(P) \cup E(Q))$ consists
of \( n (n - 3)/2 \) paths. This proves \( \gamma(K_n) \leq n(n - 3)/2 \). Next, suppose that for \( k = (n(n - 3)/2) - 1 \) there exists a graphoidal cover \( \{P_1, P_2, \ldots, P_k\} \) of \( K_n \). Then

\[
n(n - 1)/2 = |E(K_n)| \leq \sum_{i=1}^{k} |E(P_i)| = k + \sum_{i=1}^{k} t(P_i)
\]

whence

\[
\sum_{i=1}^{k} t(P_i) \geq (n(n - 1)/2) - k = n + 1.
\]

This contradicts C2, and the theorem follows.

Figure 1 displays all graphoidal covers of \( K_3 \) and the six nonisomorphic graphoidal covers of \( K_4 \) that contain no closed paths. (Note that the solid-circle vertices and "T-ends" on edges indicate path terminals while open-circle vertices indicate transits.) One may easily see that \( \gamma(K_4) = 2 \). Figure 2 displays two 'nonisomorphic' minimum graphoidal covers of \( K_5 \).

Fig. 1. Certain graphoidal covers of \( K_n \) (n=2,3,4). Fig. 2. Two nonisomorphic graphoidal covers of \( K_5 \).

In certain cases, the bounds for \( \gamma(G) \) in (2) can be bettered. In the following, we provide such an improved upper bound for \( \gamma(G) \) when \( G \) varies in a class of graphs with point connectivity 2 which admit certain labelings of their points.
Let $G = (V, E)$ be a graph with $n$ points and let $f : V \to n$ be a labeling of its points (i.e., $f$ is a bijection). Orient the lines $uv$ of $G$ from $u$ to $v$, written $u \to v$, provided that $f(u) < f(v)$. Such an orientation of $G$ is called a low-to-high orientation of $G$ with respect to (w.r.t.) the given labeling $f$. By $\uparrow G_f$ we shall mean that $G$ is a graph together with a labeling $f$ w.r.t. which the lines of $G$ are oriented low-to-high. Next, we call $f$ graphoidal if the set $\pi (\uparrow G_f)$ of all the maximal directed paths in $\uparrow G_f$ is a graphoidal cover of $G$, and if $G$ admits such a labeling $f$ of its points then $G$ is said to be label graphoidal. Some label graphoidal graphs are exhibited in Fig. 3—in each graph a graphoidal labeling is displayed and the corresponding graphoidal cover $\Psi$ is written below it in the form of a list of its member paths (as in Fig. 2).

![Graphoidal covers from labelings](image)

**Fig. 3.** Graphoidal covers from labelings.

**Lemma 1**—Suppose that a graph $G$ has a graphoidal labeling $f$. If for a point $u$ in $G$ the indegree of $u$ in $\uparrow G_f$ is nonzero then its outdegree in $\uparrow G_f$ is at most one.

**Proof:** If $id(u) > 0$ and $od(u) \geq 2$ then $u$ would be a transit of more than one path in $\pi (\uparrow G_f)$, a contradiction to the hypothesis that $f$ is graphoidal.

**Lemma 2**—Suppose that a graph $G$ has a graphoidal labeling $f$. Then any point of $G$ whose degree exceeds two must either be a sink (i.e., a point at which all the arcs incident are incoming) or a source (i.e., a point at which all the incident arcs are outgoing) in $\uparrow G_f$.

**Proof:** This is a consequence of Lemma 1.

We are now led to the following characterization of label graphoidal graphs.

**Theorem 2**—A graph $G = (V, E)$ is label graphoidal if and only if every odd cycle in $G$ has a point of degree two.
Proof: Necessity—Suppose that Z is an odd cycle in G such that each point of Z has degree at least three in G. Let f be any graphoidal labeling of G. Then by Lemma 2, we see that every point of Z is either a sink or a source in \( \uparrow G_f \). But since the length of Z is odd this turns out to be impossible (since for every arc \((u, v)\) of Z, the tail \(u\) must then be a source and the head \(v\) must be a sink so that the number of points in Z must be even being equal to twice the number of sources (sinks) in it). Hence, it follows that Z must contain a point of degree two. Since Z was an arbitrarily chosen odd cycle in G the necessity follows.

Sufficiency—If G has no odd cycles then it must be a bipartite graph. Let \( \{A, B\} \) be a bipartition of G with \(|A| = a\) and \(|B| = b\). Then any bijection \(f: V(G) \rightarrow c, c = (a + b)\) for which

\[
1 \leq f(u) \leq a \quad \forall \ u \in A,
\]

and

\[
a + 1 \leq f(v) \leq a + b \quad \forall \ v \in B
\]

is graphoidal since in \( \uparrow G_f \) every point of A is a source and every point of B is a sink so that \(\Psi = \{(u, v)\mid uv \in E(G), u \in A \text{ and } v \in B\}\) is a graphoidal cover of G.

Suppose that G has an odd cycle. By hypothesis, each odd cycle of G contains a vertex of degree two. Let S be the set of vertices in G that have degree two and lie on an odd cycle. Let T be minimal subset of S such that the graph \(H = G - T\) has no odd cycles. Observe that the minimality condition on T assures that T is an independent set of vertices; for if two vertices \(u\) and \(v\) of S are adjacent, then \(v\) lies on no cycle in the subgraph \(G - u\).

Now, since H has no odd cycles, it is bipartite. Let \(\{A, B\}\) be a bipartition of H with \(|A| = a, |B| = b\), and let \(|T| = t\). Consider any labeling \(f: V(G) \rightarrow d, d = (a + b + t)\) such that

\[1 \leq f(u) \leq a \quad \forall \ u \in A,\]

\[a + 1 \leq f(v) \leq a + t \quad \forall \ v \in T \text{ and} \]

\[a + t + 1 \leq f(w) \leq a + b + t \quad \forall \ w \in B.\]

Then in \(\uparrow G_f\), A is a set of sources and B is a set of sinks. Moreover, each vertex \(u\) in T is adjacent from one vertex of A, to one vertex of B, and with no others. Thus, each vertex of T is an internal point of just one path in \(\uparrow G_f\). Hence, \(f\) is graphoidal and G is label graphoidal. (See Fig. 4 for illustration).

Thus, by the above theorem, for any graph G in which every odd cycle has a point of degree two one has

\[\gamma(G) \leq |E(G)| - 2|T| + |T| = |E(G)| - |T| \quad \ldots (3)\]
where $T$ is a set of points of degree two constructed in the proof of the theorem.

In particular, if $G$ is a tree with $m$ points $v_1, v_2, \ldots, v_m$ of degree at least three, any graphoidal cover $\Psi$ of $G$ consists essentially of certain line-disjoint paths of $G$. The minimum number of line-disjoint paths in a tree $G$ with $n$ points of degree one equals

$$| n - \sum_{i=1}^{m} e_i - 1 |$$

where $| r |$ denotes the absolute value of the real number $r$ and $e_i$ is the number of even integers $s_i$ with $3 \leq s_i \leq d(v_i)$. Therefore, we have

$$| n - e - 1 | \leq \gamma G$$

where $e = \sum_{i=1}^{m} e_i$. This bound is sharp due to the fact that for any tree $G$ in which every point has degree at most three the bound in (4) is attained.

It is open to find a sharp lower bounded for $\gamma G$ in general.

Given a graph $G$ $(V, E)$ and a graphoid $\Psi$ on $G$, the intersection graph of $\Psi$, denoted $\Omega(\Psi)$ or $\Omega(G, \Psi)$ if we want to specify $G$, is called the graph of the graphoid $\Psi$. That is, the points of $\Omega(G, \Psi)$ are the paths in $\Psi$ with any two of them defined adjacent whenever they share a point of $G$ in common. Figure 5 displays the graph of the graphoid $\Psi_1$ of $K_5$ shown in Figure 2 while for $\Psi_2$ of that figure we have $\Omega(\Psi_2) = K_5$.

For any graph $G$, since $E(G) \in \mathcal{G}(G)$ we see that the line graph $L(G)$ of $G$ is the graph of a graphoid on $G$. Also, since $\Psi$ may be regarded as a special kind of hypergraph the notion of the graph of a graphoid is the special instance of the well known notion of the representative graph $L(H)$ of a hypergraph $H$ (see Berge, p.
Hence, a graph $G$ is said to be graphoidal if there exists a graph $H$ and $\Psi \in \mathcal{G}(H)$ such that $G = \Omega(H, \Psi)$. Thus, the well known line graphs (i.e. graphs $G$ for which there exist graphs $H$ such that $G = L(H)$ are all graphoidal. A well known characterization of line graphs due to Beineke\(^1\) classifies them as precisely those which do not contain the nine graphs of Figure 6 as induced subgraphs (thus, they are the "minimal forbidden subgraphs" of a line graph). However, each of them is graphoidal as it is the graph of (one of the several possible) the graphoid shown on its right. This leads us to the question whether all graphs are graphoidal! We believe this is true and pose the following.

Conjecture: Every graph is graphoidal.

Hence, for any graph $G$ let $\theta(G)$ denote the class of such graphs $H$ for which $G = \Omega(H, \Psi)$ for some $\Psi \in \mathcal{G}(H)$. We do not know if $\theta(G)$ is finite or infinite as such. It is clear that each graph in $\theta(G)$ must, however, have at least $p - 1$ lines where $p$ is the order of $G$. If this conjecture were true then $\theta(G)$ would be nonempty for every graph $G$ (it is nonempty if $G$ is a line graph, as mentioned above) and then

$$\mu_0(G) = \inf_{H \in \theta(G)} |V(H)|$$

and

$$\mu_1(G) = \inf_{H \in \theta(G)} |E(H)|$$

would be a graph invariant which could be called point graphoidal (or $pg -$) number and line graphoidal (or $lg -$) number of $G$, respectively. Also, further, we would have

$$\mu_0(G) \geq \pi(G)$$  \hspace{1cm} \text{(5)}

and

$$\mu_1(G) \geq p - 1$$ \hspace{1cm} \text{(6)}

for any $(p, q)$-graph $G$, where $\pi(G)$ denotes the well known intersection number of $G$ [viz. the minimum number of vertices on which there is a hypergraph $H$ such that $G$ is isomorphic to the representative graph $L(H)$ of $H$ (see Berge\(^2\), p. 400, Prop. 1)]. It
would hence be interesting to determine the graphs which satisfy equalities in (5) and (6) respectively.

If $G$ is a graph such that $\theta(G)$ contains a tree then $G$ belongs to a class of graphs well known$^3$-$^5$ by name "vertex intersection graphs of paths in a tree (shortly, VPT graphs)" or simply as "path graphs" whose characterization is known:

**Theorem 3**—$G$ is a VPT graph if and only if $G$ is triangulated and is the intersection graph of paths of a graph satisfying the Helly property.
[See Berge, p. 397, for the definition of the Helly property of a hypergraph]

However, not every VPT graph is the intersection graph of a graphoid on a tree. It is, therefore, interesting to investigate the structure of VPT graphs which are representable as the intersection graphs of graphoids on trees. This problem is quite open:

Many problems can be thought of on graphs which attain bounds in (5). We close by posing only three, assured that the interested and inquisitive readers will find more in their investigations.

Problem 1—Determine the graphs $G$, minimum graphoids on which are all isomorphic as hypergraphs. Here, by a ‘minimum’ graphoid on $G$ we mean a graphoidal cover of $G$ with minimum possible number of paths in it.

Problem 2—Determine the graphs $G$ such that for any two graphs $H, H' \in \theta(G)$ with $|V(H)| = |V(H')| = \nu_0(G)$, $\Omega(H, \Psi) = \Omega(H', \Psi') = \Psi$ and $\Psi'$ are isomorphic as hypergraphs. Such graphs may be termed as uniquely isographoidal graphs.

Problem 3—Determine the graphs $G$ such that two graphs $H, H' \in \theta(G)$ with $|V(H)| = |V(H')| = \nu_0(G)$ are isomorphic.

References

7. F. Harary, Graph Theory. Addison-Wesley, Reading, Massachusetts, 1972.