ON THE APPROXIMATION OF AN ANALYTIC FUNCTION BY
EXPONENTIAL POLYNOMIALS

A. NAUTIYAL* AND D. P. SHUKLA

Department of Mathematics, Indian Institute of Technology, Kanpur 208016

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For a function $f(s) = \sum_{n=0}^{\infty} a_n \exp (s \lambda_n)$, $s = \sigma + it$, analytic in the half
plane $\Re s < \alpha \ (-\infty < \alpha < \infty)$, let $E_k(f, \beta)$, the error in approximating the
function $f(s)$ by exponential polynomials in the half plane $\Re s \leq \beta \ (\beta < \alpha)$; be defined as

$$E_k(f, \beta) = \inf_{b_0, b_1, \ldots, b_k} \left[ \max_{-\infty < t < \infty} \left| f(\beta + it) \right| - \sum_{n=0}^{k} b_n \exp ((\beta + it) \lambda_n) \right] \quad \{ k = 0, 1, 2, \ldots \}$$

In the present paper, we have characterized the order and the type of the
function $f(s)$ in terms of the rate of decay of the approximation error $E_k(f, \beta)$.

1. INTRODUCTION

Let $\{\lambda_n\}_{n=0}^{\infty}$ be a given sequence of real numbers such that $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots$, $\lambda_n \to \infty$ as $n \to \infty$ and the following condition (1.1) is satisfied

$$\lim_{n \to \infty} \inf (\lambda_n - \lambda_{n-1}) = \delta > 0.$$  \hspace{1cm} (1.1)

Then $\{\lambda_n\}$ also satisfies

$$\lim_{n \to \infty} \sup \frac{n}{\lambda_n} = D < \infty.$$ \hspace{1cm} (1.2)

Now, consider the Dirichlet series

$$f(s) = \sum_{n=0}^{\infty} a_n \exp (s \lambda_n)$$ \hspace{1cm} (1.3)

where $s = \sigma + it$, $\sigma$ and $t$ being real variables. If the series given by (1.3) converges
absolutely in the half plane $\Re s < \alpha \ (-\infty < \alpha < \infty)$ then it is known (Mandelbrojt
1944, p. 166) that the series (1.3) represents an analytic function in $\Re s < \alpha$ and
since (1.2) is satisfied we have

$$-\alpha = \lim_{n \to \infty} \sup \frac{(\log | a_n |)}{\lambda_n}.$$ \hspace{1cm} (1.4)

*Present address: Systems Analysis Divison, Defence Research and Development Laboratory,
Kanchanbagh, Hyderabad 500258.
Let $D_\alpha$ denote the class of all functions $f(s)$, given by (1.3), which are analytic in $\Re s < \alpha (-\infty < \alpha < \infty)$. If, in (1.3), $a_n = 0$ for $n \geq k + 1$ and $a_k \neq 0$, then $f(s)$ will be called an exponential polynomial of degree $k$. The class of all exponential polynomials of degree at most $k$ will be denoted by $\pi_k$.

To study the growth of a function $f(s) \in D_\alpha$, the concept of order $\rho$ (Juneja and Nandan 1978) is defined as

$$\rho \equiv \rho(f) = \limsup_{\sigma \to \alpha} \frac{\log \log M(\sigma, f)}{-\log(1 - \exp(\sigma - \alpha))} \ldots (1.5)$$

where $M(\sigma, f) \equiv M(\sigma) = \max_{-\infty < t < \infty} |f(\sigma + it)|$; and if $0 < \rho < \infty$, the type $T$ of $f(s)$ is defined as

$$T \equiv T(f) = \limsup_{\sigma \to \alpha} \frac{\log M(\sigma, f)}{(1 - \exp(\sigma - \alpha))^{1/\rho}} \ldots (1.6)$$

Let $\overline{D}_\beta$, $-\infty < \beta < \infty$, be the class of all functions $f(s)$, given by (1.3), analytic in $\Re s \leq \beta$, i.e., $f(s) \approx \overline{D}_\beta$ if $f(s) \in D_{\alpha_0}$ for some $\alpha_0 > \beta$. For $f(s) \in \overline{D}_\beta$, we define $E_n(f, \beta)$, the error in approximating the function $f(s)$ by exponential polynomials of degree $n$ in uniform norm as

$$E_n(f, \beta) = \inf_{p \in \pi_n} \|f - p\|_{\beta}, \ n = 0, 1, 2, \ldots \ldots (1.7)$$

where

$$\|f - p\|_{\beta} = \max_{-\infty < t < \infty} |f(\beta + it) - p + (\beta + it)|.$$

Juneja and Nandan (1978) have studied order $\rho$ and type $T$ of $f(s) \in D_\alpha$ in terms of the coefficients $a_n$'s in (1.3). In the present paper we have characterized order $\rho$ and type $T$ of $f(s) \in D_\alpha$ in terms of the rate of decay of the approximation error $E_n(f, \beta), \beta < \alpha$, as $n \to \infty$.

We prove:

**Theorem 1**—Let $f(s) \in \overline{D}_\beta$, $-\infty < \beta < \infty$. Then $f(s) \in D_\alpha$, $\beta < \alpha < \infty$, if and only if,

$$\limsup_{n \to \infty} \frac{\log E_n(f, \beta)}{\lambda_{n+1}} = \beta - \alpha. \ldots (1.8)$$

**Theorem 2**—Let $f(s) \in D_\alpha$ be of order $\rho$ and $-\infty < \beta < \alpha < \infty$. Then

$$\rho = \limsup_{n \to \infty} \frac{\log^+ \log^+ E_n(f, \beta) \exp((\alpha - \beta) \lambda_{n+1})}{\log \lambda_{n+1} - \log^+ \log^+ (E_n(f, \beta) \exp((\alpha - \beta) \lambda_{n+1}))} \ldots (1.9)$$

where $\log^+ x = \max(0, \log x)$.

**Theorem 3**—Let $f(s) \in D_\alpha$ and $-\infty < \beta < \alpha < \infty$. Then $f(s)$ is of order $\rho$ ($0 < \rho < \infty$) and type $T$, if and only if,

$$\nu = \frac{(-1)^{\rho+1}}{\rho^\nu} T$$
where
\[ v = \limsup_{n \to \infty} \frac{\log^+ (E_n (f, \beta) \exp ((\alpha - \beta) \lambda_{n+1}))^{p+1}}{\lambda_{n+1}^p} \] ... (1.10)

satisfies \( 0 < v < \infty \).

2. Preparatory Lemmas

In this section we give some lemmas required in the proofs of Theorems 1, 2 and 3.

Lemma 1—Let \( f(s) \in D_\alpha \) and \( -\infty < \beta < \alpha < \infty \). Then, for all \( \sigma (\sigma < \alpha) \) sufficiently close to \( \alpha \), we have

\[ E_k (f, \beta) \leq KM (\sigma, f)/\exp ((\sigma - \beta) \lambda_{k+1}), \quad k = 0, 1, 2, \ldots \]

where \( K \) is a constant independent of \( k \) and \( \sigma \).

Proof: Let \( f(s) \in D_\alpha \) be given by (1.3) and let

\[ p_k (s) = \sum_{n=0}^{k} a_n \exp (s \lambda_n) \]

be the \((k + 1)\)th partial sum of the series (1.3) of \( f(s) \). Now using the definition (1.7) of \( E_k (f, \beta) \) we have

\[ E_k (f, \beta) \leq \|f - p_k\|_{\beta} \leq \sum_{n=k+1}^{\infty} |a_n| \exp (\beta \lambda_n) \leq M(\sigma) \sum_{n=k+1}^{\infty} \exp ((\beta - \sigma) \lambda_n) \] ... (2.1)

for \( \sigma > \beta \), since by Cauchy's inequality \( |a_n| \leq M (\sigma)/\exp (\sigma \lambda_n) \) for all \( n \) and all \( \sigma < \alpha \). Further, since (1.1) is satisfied, we can choose \( 0 < \delta' < \delta \) such that \( (\lambda_n - \lambda_{n-1}) \geq \delta' \) for all \( n \geq 0 \). Thus, for \( \sigma \geq (\alpha + \beta)/2 \), (2.1) gives that

\[ E_k (f, \beta) \leq M(\sigma) \exp ((\beta - \sigma) \lambda_{k+1}) \sum_{n=k+1}^{\infty} \exp ((\beta - \sigma) (\lambda_n - \lambda_{k+1})) \]

\[ \leq M(\sigma) \exp ((\beta - \sigma) \lambda_{k+1}) \sum_{n=0}^{\infty} \exp (- (\alpha - \beta) \delta' n/2) \]

\[ = M(\sigma) \exp ((\beta - \sigma) \lambda_{k+1}) \frac{1}{1 - \exp ((\beta - \alpha) \delta'/2)}. \]

The lemma follows from the above inequality.

Lemma 2—Let \( f(s) \in \tilde{D}_\beta \), \( -\infty < \beta < \infty \), be given by (1.3). Then for \( n \geq 1 \), we have

\[ |a_n| \exp (\beta \lambda_n) \leq 2 E_{n-1} (f, \beta). \]

Proof: For \( f(s) \in \tilde{D}_\beta \) we have (Mandelbrojt 1944, Theorem IX)
\[ a_n \exp(\beta \lambda_n) = \lim_{t_0 \to \infty} \frac{1}{t_*} \int_{t_0}^{t_*} f(\beta + it) \exp(-i t \lambda_n) \, dt. \tag{2.2} \]

Further, if \( x \) is real and \( x \neq 0 \), then
\[ \lim_{t_0 \to \infty} \frac{1}{t_*} \int_{t_0}^{t_*} \exp(x(\beta + it)) \, dt = 0. \tag{2.3} \]

From (2.2) and (2.3) we get
\[ a_n \exp(\beta \lambda_n) = \lim_{t_0 \to \infty} \frac{1}{t_*} \int_{t_0}^{t_*} \left( f(\beta + it) - p(\beta + it) \right) \exp(-it \lambda_n) \, dt \]

for any \( p(s) \in \pi_{n-1} \). The above relation easily gives that
\[ |a_n| \exp(\beta \lambda_n) \leq \|f - p\|_\beta \tag{2.4} \]

for any \( p(s) \in \pi_{n-1} \). By the definition (1.7) of \( E_n(f, \beta) \) there exists \( \tilde{p}(s) \in \pi_{n-1} \) such that
\[ \|f - \tilde{p}\|_\beta \leq 2 \, E_{n-1}(f, \beta). \tag{2.5} \]

Taking, in particular \( p(s) = \tilde{p}(s) \) in (2.4) and using (2.5) the lemma follows from (2.4).

We also need the following coefficient characterizations, obtained by Juneja and Nandan (1978), of order \( \rho \) and type \( T \) of \( f(s) \in D_\alpha \) given by (1.3).

**Lemma 3**—Let \( f(s) \in D_\alpha \), given by (1.3), be of order \( \rho \). Then
\[ \rho = \limsup_{n \to \infty} \frac{\log^+ \log^+(|a_n| \exp(\alpha \lambda_n))}{\log \lambda_n - \log^+ \log^+(|a_n| \exp(\alpha \lambda_n))}. \]

**Lemma 4**—Let \( f(s) \in D_\alpha \), given by (1.3), be of order \( \rho \) (\( 0 < \rho < \infty \)) and type \( T \). Then
\[ T(\rho + 1)^{\rho+1}/\rho^\rho = \limsup_{n \to \infty} \frac{\log^+(|a_n| \exp(\alpha \lambda_n))^{\rho+1}}{\lambda_n^\rho}. \]

**Remark**: In Juneja and Nandan (1978), Lemmas 3 and 4 were obtained with the condition (1.2) on the exponents \( \lambda_n^\nu \)'s instead of (1.1).

### 3. **Proofs of the Theorems**

**Proof of Theorem 1**—First suppose that \( f(s) \in D_\beta \) belongs to \( D_\alpha \), \( \beta \leq \alpha < \infty \). Then, by Lemma 1, we have
\[
\limsup_{n \to \infty} (\log E_n (f, \beta))/\lambda_{n+1} \leq \beta - \sigma
\]
for all \( \sigma \) sufficiently close to \( \alpha \) and so
\[
\limsup_{n \to \infty} (\log E_n (f, \beta))/\lambda_{n+1} \leq \beta - \alpha. \tag{3.1}
\]

On the other hand, using Lemma 2 and (1.4), since \( f(s) \in D_\alpha \), we have
\[
- \alpha = \limsup_{n \to \infty} (\log |a_n|)/\lambda_n \leq - \beta + \limsup_{n \to \infty} (\log E_n (f, \beta))/\lambda_{n+1}. \tag{3.2}
\]

The necessity part of the theorem follows from (3.1) and (3.2).

Sufficiency part can also be proved similarly. This proves the theorem.

**Proof of Theorem 2**—Let the limit superior on the right hand side of (1.9) be denoted by \( d \). Obviously \( 0 \leq d \leq \infty \). First, let \( 0 < d < \infty \) and \( 0 < d' < d \). Then, there exists a sequence \( \{n_k\} \) of positive integers tending to \( \infty \) such that
\[
\log E_{n_k} (f, \beta) + \lambda_{n_k+1} (\alpha - \beta) > (\lambda_{n_k+1})^{d'/1+\beta} \tag{3.3}
\]
for \( k = 1, 2, 3, \ldots \). Now, using Lemma 1, (3.3) gives that
\[
\log M(\sigma, f) \geq (\lambda_{n_k+1})^{d'/1+\beta} + \lambda_{n_k+1} (\sigma - \alpha) - \log K \tag{3.4}
\]
for the sequence \( \{n_k\} \) and all \( \sigma (\sigma < \alpha) \) sufficiently close to \( \alpha \). Let \( \{a_k\} \) be the sequence defined as
\[
\sigma_k = \alpha - (d'/1+\beta)(1/\lambda_{n_k+1})^{1/(d'+1)}.
\]
Then \( \sigma_k \to \alpha \) as \( k \to \infty \). Now, using (3.4), for all sufficiently large values of \( k \) we have
\[
\log M(\sigma_k, f) \geq \frac{(d')^{d'}}{(1+d')^{1+d'}} (\alpha - \sigma_k)^{-d'} - \log K.
\]
Since \( 1 - \exp(\sigma_k - \alpha) \sim (\alpha - \sigma_k) \) as \( k \to \infty \), the above inequality gives that
\[
\limsup_{k \to \infty} \log \log M(\sigma_k, f) \geq \frac{(d')^{d'}}{(1+d')^{1+d'}} \tag{3.5}
\]
and, since \( d' (d) \) is arbitrary, this gives
\[
\rho \geq d. \tag{3.5}
\]
Obviously, (3.5) holds for \( d = 0 \). For \( d = \infty \) the above analysis that \( \rho = \infty \).

On the other hand, using (1.3) and Lemma 2, for \( f(s) \in D_\alpha \), we get
\[
M(\sigma, f) \leq \sum_{n=0}^{\infty} |a_n| \exp(\sigma \lambda_n) \leq |a_0| + 2 \sum_{n=1}^{\infty} E_{n-1} (f, \beta) \exp((\sigma - \beta) \lambda_n) = |a_0| + 2M(\sigma, f_\beta) \tag{3.6}
\]
where, by Theorem 1,
\[ f_\beta(s) = \sum_{n=1}^{\infty} \{E_{n-1} (f, \beta) \exp (-\beta \lambda_n)\} \exp (s \lambda_n) \quad \ldots (3.7) \]

belongs to \( D_\alpha \). From (3.6) we get \( \rho \leq \rho (f_\beta) \), where \( \rho (f_\beta) \) is the order of \( f_\beta(s) \).

Applying Lemma 3 to \( f_\beta(s) \), we now get

\[ \rho \leq d \quad \ldots (3.8) \]

Theorem now follows from (3.5) and (3.8).

**Proof of Theorem 3**—First suppose that \( f(s) \) is of order \( \rho \) and type \( T, \ T < \infty \).

Then, given \( \epsilon > 0 \), (1.6) gives that there exists \( \sigma_0 = \sigma_0(\epsilon) \) such that

\[ \log M(\sigma, f) \leq (T + \epsilon) (1 - \exp (\tau - \alpha))^{-\rho} \]

for \( \sigma_0 < \sigma < \alpha \). Using Lemma 1 this gives

\[ \log^+ (E_n (f, \beta) \exp ((\alpha - \beta) \lambda_n)) \leq (T + \epsilon) (1 - \exp (\sigma - \alpha))^{-\rho} + \lambda_{n+1} (\alpha - \sigma) + \log^+ K \quad \ldots (3.9) \]

for all \( n \) and all \( \sigma \) sufficiently close to \( \alpha \). Chose a sequence \( \{\sigma_n\} \) as

\[ (1 - \exp (\sigma_n - \alpha)) = ((T + \epsilon) \rho / \lambda_{n+1})^{1/(p+1)}. \quad \ldots (3.10) \]

Clearly \( \sigma_n \to \alpha \) as \( n \to \infty \). Using (3.9) and (3.10) we get

\[ \log^+ (E_n (f, \beta) \exp ((\alpha - \beta) \lambda_{n+1})) \leq \frac{(T + \epsilon)^{1/(p+1)} \lambda_{n+1}^{p/(p+1)}}{\rho p/(1 + p)} \times (1 + \rho + o(1)) \]

for all sufficiently large values of \( n \). This, on proceeding to limits easily gives

\[ \nu \leq (\rho + 1)^{p+1} T/\rho^p. \quad \ldots (3.11) \]

On the other hand, it follows from Theorem 2 and Lemma 3 that the order of \( f_\beta (s) \),

given by (3.7), is equal to the order \( \rho \) of \( f(s) \). Using (3.6) we now get \( T \leq T(f_\beta) \),

where \( T(f_\beta) \) is the type of \( f_\beta(s) \) and so, applying Lemma 4 to \( f_\beta(s) \), we obtain

\[ (\rho + 1)^{p+1} T/\rho^p \leq \nu. \quad \ldots (3.12) \]

Necessity part of the theorem now follows from (3.11) and (3.12).

Conversely, suppose that \( 0 < \nu < \infty \), then (1.10) easily gives

\[ \rho = \limsup_{n \to \infty} \frac{\log^+ \log^+ (E_n (f, \beta) \exp ((\alpha - \beta) \lambda_{n+1}))}{\log \lambda_{n+1} - \log^+ \log^+ (E_n (f, \beta) \exp ((\alpha - \beta) \lambda_{n+1}))} \]

and so, by Theorem 2, \( f(s) \) is of order \( \rho \). Sufficiency part of the theorem now follows from the necessity part.

This proves the theorem.

**References**
