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ADJOINTNESS OF PARTIAL DIFFERENTIAL OPERATOR IN CURVILINEAR COORDINATES

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The curvilinear coordinates has been employed to verify adjointness of simple partial differential operator, Sturm-Liouville operator and Laplacian operator in framework of the realizability technique.

1. INTRODUCTION

In a set of previous papers (Morsy and Ata 1971a, b, c) ordinary differential operator of order *n* is shown not to be self-adjoint one. In this respect it is of importance to emphasize that non self-adjointness has its impact both on the concept of completeness (Morsy and Ata 1971a, c, 1975) as well as on the identity of differential operators having its essential grounds for quantum mechanics. Non self-adjointness of a differential operator is partially associated with the presence of those non-vanishing boundary terms arising at extracting the adjoint which can be obtained through approaches requiring integration by parts. Realization of self-adjointness for operators leads to consistency with the invariance property (Sokolinkoff 1951) thus proper mathematical formulation being therefore achieved.

The curvilinear coordinates have been employed to extract the adjoint of first order differential operator (Gruber 1972). This was accomplished by a trial to select special basis that gives rise possibility for vanishing of boundary terms. On such grounds self-adjointness still non-realized, it is therefore our view point to study such an interesting problem aiming to restore self-adjointness.

This article is essentially devoted to establish methodological approaches instead of that traditional ones to discuss the self-adjointness in curvilinear coordinates. In such view points, it is of importance to reveal that realizability technique has been proposed to facilitate verifying the self-adjointness irrespective of missing proper spaces needed for representations (Morsy and Ata 1971a, b). In this respect and to promote the representations, our technique is tried in framework of curvilinear coordinates for hermitizing the partial differential operator of any order. A wide scope for both the classical and the quantal description has therefore been achieved.

2. Adjointness of the Differential Operators in Curvilinear Coordinates

2.1. *Adjoint of Partial Differential Operator*

Let us first start to study self-adjointness of such a partial differential operator

$$\mathsf{D}^n_{q_i} \equiv \frac{\partial^n}{\partial^n q_i} \qquad \text{...(2.1.1)}$$

in curvilinear coordinates proceeding as

$$\langle \Psi \mid \mathsf{D}^n_{q_i} \mid \phi \rangle = \int_D d^3q \sqrt{g(q)}\ \Psi^*(q)\, \mathsf{D}^n_{q_i}\, \phi(q), \quad i = 1, 2, 3 \qquad \text{...(2.1.2)}$$

where

$$\int_D d^3q \equiv \int dq_1 \int dq_2 \int dq_3$$

stands for the volume integral operator. Here $g(q)$ is the determinant of the metric tensor. The functions ϕ and Ψ with their derivatives possess continuity requirements via the variant characterising the operator. Introducing the right handed operator $\mathsf{a}^n_{q_i}$ (Morsy and Ata 1971a) which acts on the operand placed to its left, we get

$$\langle \Psi \mid \mathsf{D}^n_{q_i} \mid \phi \rangle = \left\langle \phi \left| \frac{1}{\sqrt{g}}\ \mathsf{a}^n_{q_i}\ \sqrt{g} \right| \Psi \right\rangle^* \qquad \text{...(2.1.3)}$$

thus having

$$\mathsf{D}^{n\dagger}_{q_i} = \frac{1}{\sqrt{g}}\ \mathsf{a}^n_{q_i}\ \sqrt{g}. \qquad \text{...(2.1.4)}$$

The relation governing the concerned right and left handed operators being arised by considering the integration by parts and by making use of mathematical induction leading to

$$\mathsf{a}^n_{q_i} = (-1)^n\ \mathsf{D}^n_{q_i} + (-1)^{\nu-1}\ \mathsf{a}^{n-\nu}_{q_i}\ \tilde{\delta}_{q_i}\ \mathsf{D}^{\nu-1}_{q_i}, \quad \nu = 1, \ldots, n \qquad \text{...(2.1.5)}$$

the symbol $\tilde{\delta}_{q_i}$ denotes

$$\tilde{\delta}_{q_i}\ (q_i, b_i, a_i) = \delta(q_i - b_i) - \delta(q_i - a_i),$$

in which $\delta(q_i - b_i)$, $\delta(q_i - a_i)$ are the Dirac delta functions (Messiah 1961) $\forall q_i \in [a_i, b_i]$. Here a_i and b_i specify the concerned boundary value problem.

It is then possible to obtain the adjoint of our operator as

$$\mathsf{D}^{n\dagger}_{q_i} = \frac{1}{\sqrt{g}} [(-1)^n \mathsf{D}^n_{q_i} + (-1)^{\nu-1} \mathfrak{a}^{n-\nu}_{q_i} \tilde{\delta}_{q_i} \mathsf{D}^{\nu-1}_{q_i}] \sqrt{g}, \quad \nu = 1, \ldots, n. \qquad \ldots(2.1.6)$$

2.2. *Adjoint of Sturm-Liouville Operator*

The Sturm-Liouville operator of the second order is given as (Morse and Feshbach 1953)

$$\mathsf{E}_1 = \mathsf{D}_{q_i} r_1(q) \mathsf{D}_{q_i} + S(q) \qquad \ldots(2.2.1)$$

where $r_1(q) \equiv r_1(q_1, q_2, q_3)$ and $S(q) \equiv S(q_1, q_2, q_3)$ are real, positive continuous functions.

In matrix formulation the first term of the operator gives

$$\langle \Psi \mid \mathsf{D}_{q_i} r_1 \mathsf{D}_{q_i} \mid \phi \rangle = \int_D d^3q \sqrt{g} \Psi^* \mathsf{D}_{q_i} r_1 \mathsf{D}_{q_i} \phi$$

Integrating by parts successively to get

$$\langle \Psi \mid \mathsf{D}_{q_i} r_1 \mathsf{D}_{q_i} \mid \phi \rangle = \langle \phi \mid \mathsf{D}_{q_i} r_1 \mathsf{D}_{q_i} + \mathfrak{a}_{q_i} r_1 \tilde{\delta}_{q_i} - \tilde{\delta}_{q_i} r_1 \mathsf{D}_{q_i} \mid \Psi \rangle^*.$$

The adjoint then reads

$$\mathsf{E}^\dagger_1 = \frac{1}{\sqrt{g}} [\mathsf{E}_1 + \mathfrak{a}_{q_i} r_1 \tilde{\delta}_{q_i} - \tilde{\delta}_{q_i} r_1 \mathsf{D}_{q_i}] \sqrt{g}. \qquad \ldots(2.2.2)$$

Our treatment can be accomplished by carrying successively the integration by parts required for the concerned problem; then after mathematical induction leads to the result for the n-order operator as

$$\mathsf{E}^\dagger_n = \frac{1}{\sqrt{g}} [\mathsf{E}_n + \mathfrak{a}^{n+\mu-\nu}_{q_i} (-1)^{\nu-1} . C^n_\mu \; C^{\nu-1}_\epsilon \hat{\delta}_{q_i} r^{(n-\mu+\nu-\epsilon-1)} \mathsf{D}^\epsilon_{q_i}] \sqrt{g}, \qquad \ldots(2.2.3)$$

$$\mu = 0, \ldots, n, \quad \nu = 1, \ldots, n + \mu, \quad \epsilon = 0, \ldots, \nu - 1.$$

2.3. *Adjoint of the Laplacian Operator*

Before investigating the adjoint of the Laplacian operator, it is first useful to study the adjoint of the gradient operator of such a form

$$\bar{\nabla} \equiv \bar{e}_i \frac{1}{\sqrt{g_{ii}}} \mathsf{D}_{q_i} = \frac{\bar{e}_i}{\sqrt{g_{ii}}} \mathsf{D}_{q_i} \qquad \ldots(2.3.1)$$

described in the curvilinear coordinates.

In matrix notations it reads

$$\langle \Psi \mid \bar{\nabla} \mid \phi \rangle = \bar{e}_i \int_D d^3q\sqrt{g}\phi \frac{1}{\sqrt{g}} \mathrm{Q}_{q_i} \sqrt{\frac{g}{g_{ii}}} \Psi^*$$

$$= \bar{e}_i \left\langle \phi \left| \frac{1}{\sqrt{g}} \mathrm{Q}_{q_i} \sqrt{\frac{g}{g_{ii}}} \right| \Psi \right\rangle^*. \quad \text{...(2.3.2)}$$

Hence the adjoint of the gradient is

$$\bar{\nabla}^\dagger = \bar{e}_i \frac{1}{\sqrt{g}} \mathrm{Q}_{q_i} \sqrt{\frac{g}{g_{ii}}}. \quad \text{...(2.3.3)}$$

Making use of the relation (2.1.5) governing left and right handed operators to get.

$$\bar{\nabla}^\dagger = \frac{-1}{\sqrt{g}} \bar{\nabla}\sqrt{g} + \bar{e}_i(\tilde{\delta}_{q_i} - \mathrm{D}_{q_i}) \frac{1}{\sqrt{g_{ii}}}. \quad \text{...(2.3.4)}$$

For the Laplacian operator let us start with the form

$$\nabla^2 \equiv \frac{1}{\sqrt{g}} \mathrm{D}_{q_i} \sqrt{g}\, g^{ij}\, \mathrm{D}_{q_j} \quad \text{...(2.3.5)}$$

where g^{ij} is the reciprocal fundamental tensor (assumed to be diagonal). Then having

$$\langle \Psi \mid \nabla^2 \mid \phi \rangle = \int_D d^3q\sqrt{g}\Psi^* \frac{1}{\sqrt{g}} \mathrm{D}_{q_i} \sqrt{g}\, g^{ii}\, \mathrm{D}_{q_i}\phi$$

and taking the transpose of the above integral to get

$$\langle \Psi \mid \nabla^2 \mid \phi \rangle = \left\langle \phi \left| \frac{1}{\sqrt{g}} \mathrm{Q}_{q_i} g^{ii} \sqrt{g} \mathrm{Q}_{q_i} \right| \Psi \right\rangle^*. \quad \text{...(2.3.6)}$$

Making use of eqn. (2.1.5), the adjoint of the Laplacian then reads

$$\nabla^{2\dagger} = \nabla^2 + \frac{1}{\sqrt{g}} (\mathrm{Q}_{q_i} \sqrt{g}\, g^{ii}\, \tilde{\delta}_{q_i} - \tilde{\delta}_{q_i} g^{ii} \sqrt{g} \mathrm{D}_{q_i}). \quad \text{...(2.3.7)}$$

3. Realizability Technique

It is of interest at this point to illustrate the essential role of realizability technique in realizing self-adjointness of partial differential operators.

3.1. *Realizability of Simple Partial Differential Operator*

Let us to illustrate our technique starting with first and second order partial differential operators.

Consider a relation such that

$$A_1 = a_1 \mathrm{D}_{q_i} + b_1 \frac{\sqrt{g^{(q_i)}}}{\sqrt{g}} + c_1 \tilde{\delta}_{q_i} \quad \text{...(3.1.1)}$$

describing the first order partial differential operator, here a_1, b_1 and c_1 are arbitrary constants being complex in general. Here

$$\sqrt{g^{(q_i)}} \equiv \frac{\partial}{\partial q_i} \sqrt{g}.$$

In adjoint form one has

$$A_1^\dagger = a_1^* \ \mathsf{D}_{q_i}^\dagger + b_1^* \ \frac{\sqrt{g^{(q_i)}}}{\sqrt{g}} + c_1^* \tilde{\delta}_{q_i}.$$

The self-adjointness of A_1 requires that

$$\left.\begin{aligned} a_1 &= i \ \mathrm{Im}\ a_1 \\ \mathrm{Im}\ b_1 &= \tfrac{1}{2} \ \mathrm{Im}\ a_1 \\ \mathrm{Im}\ c_1 &= -\tfrac{1}{2} \ \mathrm{Im}\ a_1. \end{aligned}\right\} \qquad \text{...(3.1.2)}$$

Our operator then reads

$$A_1 = i \ \mathrm{Im}\ a_1 \ \mathsf{D}_{q_i} + \left(\mathrm{Re}\ b_1 + \frac{i}{2} \ \mathrm{Im}\ a_1\right) \frac{\sqrt{g^{(q_i)}}}{\sqrt{g}}$$
$$+ \left(\mathrm{Re}\ c_1 - \frac{i}{2} \ \mathrm{Im}\ a_1 \right) \tilde{\delta}_{q_i}. \qquad \text{...(3.1.3)}$$

For the second order differential operator let us start with

$$A_2 = a_1 \mathsf{D}_{q_i}^2 + a_2 \mathfrak{O}_{q_i} \tilde{\delta}_{q_i} + a_3 \tilde{\delta}_{q_i} \mathsf{D}_{q_i} + b_1 \frac{\sqrt{g^{(q_i)}}}{\sqrt{g}} \mathsf{D}_{q_i}$$
$$+ b_2 \frac{1}{\sqrt{g}} \mathsf{D}_{q_i} \sqrt{g^{(q_i)}} + b_3 \tilde{\delta}_{q_i} \frac{\sqrt{g}}{\sqrt{g^{(q_i)}}} + b_4 \tilde{\delta}_{q_i} \frac{\sqrt{g^{(q_i)}}}{\sqrt{g}}. \qquad \text{...(3.1.4)}$$

In the framework of our approach, self-adjointness of A_2 requires it in a restricted form as

$$A_2 = a_1 \mathsf{D}_{q_i}^2 + a_2 \mathfrak{O}_{q_i} \tilde{\delta}_{q_i} + (a_2^* - a_1) \tilde{\delta}_{q_i} \mathsf{D}_{q_i} + b_1 \frac{\sqrt{g^{(q_i)}}}{\sqrt{g}} \mathsf{D}_{q_i}$$
$$+ (a_1 - b_1^*) \frac{1}{\sqrt{g}} \mathsf{D}_{q_i} \sqrt{g^{(q_i)}} + (b_3^* + a_2) \tilde{\delta}_{q_i} \frac{\sqrt{g}}{\sqrt{g^{(q_i)}}}$$
$$+ (b_4^* - b_1 - b_1^* + a_2^*) \tilde{\delta}_{q_i} \frac{\sqrt{g^{(q_i)}}}{\sqrt{g}}. \qquad \text{...(3.1.5)}$$

3.2. *Realizable Sturm-Liouville Operator*

Let us proceed further to consider the following operator:

$$\hat{\mathsf{E}}_1 = a_1 \mathsf{E}_1 + a_2 \mathfrak{O}_{q_i} \tilde{\delta}_{q_i} r + a_3 r \tilde{\delta}_{q_i} \mathsf{D}_{q_i} + b_1 r \frac{\sqrt{g^{(q_i)}}}{\sqrt{g}} \mathsf{D}_{q_i}$$
$$+ \frac{b_2}{g} \mathsf{D}_{q_i} r \sqrt{g^{(q_i)}} + b_3 \frac{\sqrt{g}}{\sqrt{g^{(q_i)}}} \tilde{\delta}_{q_i} r + b_4 r \tilde{\delta}_{q_i} \frac{\sqrt{g^{(q_i)}}}{\sqrt{g}}. \qquad \text{...(3.2.1)}$$

Here the adjoint reads

$$\hat{\mathsf{E}}_1^\dagger = a_1^* \mathsf{E}_1 + (a_1^* + a_3^*) \,\mathsf{D}_{q_i} \tilde{\delta}_{q_i} r + (a_3^* - a_1^*) r \tilde{\delta}_{q_i} \mathsf{D}_{q_i}$$

$$+ (a_1^* - b_2^*) \frac{\sqrt{g^{(q_i)}}}{\sqrt{g}} r \mathsf{D}_{q_i} + (a_1^* - b_1^*) \frac{1}{\sqrt{g}} \mathsf{D}_{q_i} r \sqrt{g^{(q_i)}}$$

$$+ (a_1^* + a_3^* + b_3^*) \tilde{\delta}_{q_i} r \frac{\sqrt{g}}{\sqrt{g^{(q_i)}}} + (b_1^* + b_2^* + b_4^*$$

$$+ a_2^* - a_1^*) \tilde{\delta}_{q_i} r \frac{\sqrt{g^{(q_i)}}}{\sqrt{g}}.$$

Satisfying requirements of self-adjointness, our adopted approach then realizes that

$$\hat{\mathsf{E}}_1 = a_1 \mathsf{E}_1 + a_2 \,\mathsf{D}_{q_i} \tilde{\delta}_{q_i} r + (a_2^* - a_1) r \tilde{\delta}_{q_i} \mathsf{D}_{q_i}$$

$$+ b_1 r \frac{\sqrt{g^{(q_i)}}}{\sqrt{g}} \mathsf{D}_{q_i} + (a_1 - b_1^*) \frac{1}{\sqrt{g}} \mathsf{D}_{q_i} \sqrt{g^{(q_i)}}\, r$$

$$+ (b_3^* + a_2) \tilde{\delta}_{q_i} r \frac{\sqrt{g}}{\sqrt{g^{(q_i)}}} + (b_4^* + b_1^* - b_1 + a_2^*) \delta_{q_i} r \frac{\sqrt{g^{(q_i)}}}{\sqrt{g}}. \quad \ldots(3.2.2)$$

3.3. *Realizable Laplacian Operator*

Let us first verify realizability of the gradient operator by considering

$$S_1 = \alpha \bar{\nabla} + \bar{e}_j \left[\frac{\beta}{g} \left(\sqrt{\frac{g}{g_{jj}}} \right)^{(q_i)} + \frac{\gamma}{\sqrt{g_{jj}}} \tilde{\delta}_{q_j} \right] \quad \ldots(3.3.1)$$

Our treatment leads to have

$$\left.\begin{aligned} \alpha &= -i \,\mathrm{Im}\, \alpha \\ \mathrm{Im}\; \beta &= -\tfrac{1}{2}\, \mathrm{Im}\, \alpha \\ \mathrm{Im}\; \gamma &= \tfrac{1}{2}\, \mathrm{Im}\, \alpha. \end{aligned}\right\} \quad \ldots(3.3.2)$$

Thus getting

$$S_1 = -i \,\mathrm{Im}\, \alpha \bar{\nabla} + \bar{e}_j \left(\mathrm{Re}\, \beta - \frac{i}{2}\, \mathrm{Im}\, \alpha \right) \frac{1}{\sqrt{g}} \left(\sqrt{\frac{g}{g_{jj}}} \right)^{(q_i)}$$

$$+ \left(\mathrm{Re}\, \gamma + \frac{i}{2}\, \mathrm{Im}\, \alpha \right) \frac{\tilde{\delta}_{q_i}}{\sqrt{g_{jj}}}. \quad \ldots(3.3.3)$$

Thereafter let us verify realizability of the Laplacian operator through consideration of

$$S_2 = \alpha \nabla^2 + \beta \frac{1}{\sqrt{g}} \,\mathsf{D}_{q_i} \sqrt{g}\, g^{jj} \tilde{\delta}_{q_i} + \gamma \tilde{\delta}_{q_i} g^{jj} \mathsf{D}_{q_i}. \quad \ldots(3.3.4)$$

Self-adjointness for S_2 requires that

$$\left.\begin{aligned} \alpha &= \mathrm{Re}\,\alpha \\ \beta &= \mathrm{Re}\,\alpha + \gamma^*. \end{aligned}\right\} \qquad \ldots(3.3.5)$$

Realizability of the Laplacian operator makes our operator being self-adjoint as

$$S_2 = \mathrm{Re}\,\alpha\nabla^2 + (\mathrm{Re}\,\alpha + \gamma^*)\,\frac{1}{\sqrt{g}}\,\partial_{q_i}\,\sqrt{g}\,g^{ij}\,\tilde{\partial}_{q_i} + \gamma\tilde{\partial}_{q_i}\,g^{ij}\,D_{q_j}. \qquad \ldots(3.3.6)$$

4. Discussion and Conclusion

As a conclusion the success of realizability technique in restoring self-adjointness for differential operators presents essential grounds to mathematical description for theoretical physics. As the Laplacian operator relates the quantal correspondence of the kinetic energy our approaches then make it possible to reformulate Schrodinger equation in its modified form. Possession of hermiticity property of the Hamiltonian operator admits possibility of saving integrity of the quantum theory. Having such grounds the invariance property required for proper description of physical laws is therefore being respected. A wide scope for both the classical and the quantal description has therefore been achieved.

References

Gruber, G. R. (1972). Quantization in generalized coordinates. *Int. J. Theor. Phys.*, **6**, 31.

Messiah, Albert (1961). Quantum Mechanics. North-Holland Publishing Company, Amsterdam.

Morse, P. M., and Feshbach, M. (1953). Methods of Theoretical Physics. McGraw-Hill Book Company, Inc., New York.

Morsy, M. W., and Ata, M. S. (1971a). Realizability of linear differential operators and the dispersive behaviour of scattering. *ATKE*, **17**, 203.

——— (1971b). The direct interaction theory and the non-hermiticity of the Laplacian operator. *ATKE*, **17**, 271.

——— (1971c). Realizability of the generalized Sturm-Liouville differential operators and completeness. *ATKE*, **18**, 79.

——— (1975). The unified theory of nuclear reactions and the non-hermiticity of the kinetic energy operator. *First Conference on Nuclear Science and Applications, Cairo*.

Sokolinkoff, I. S. (1951). Tensor Analysis. Appl. Math. Series. John Wiley & Sons, New York.