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ON INFINITESIMAL PROJECTIVE VARIATIONS OF A *P*-SASAKIAN HYPERSURFACE IN A LOCALLY PRODUCT RIEMANNIAN MANIFOLD

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In this paper, we have studied infinitesimal projective variations of a *P*-Sasakian hypersurface in a locally product Riemannian manifold. We have mainly proved that an infinitesimal projective variation satisfying a certain condition of a compact orientable *P*-Sasakian hypersurface is tangential and the variation leaves the *P*-Sasakian structure tensors invariant. Also, there does not exist an infinitesimal projective normal variation of a *P*-Sasakian hypersurface except a trivial variation.

1. INTRODUCTION

In the previous paper (Matsumoto 1980), we have considered an infinitesimal conformal variation of a *P*-Sasakian hypersurface in a locally product Riemannian manifold and we have gotten an interesting result about such variation.

In this paper we shall consider infinitesimal projective variations defined by Yano (1978) of a hypersurfaces in a locally product Riemannian manifold and we shall mainly prove the following:

Theorem — If an infinitesimal projective variation of a compact orientable *P*-Sasakian hypersurface in a locally product Riemannian manifold satisfies the condition (2.30), then the variation is tangential and it preserves the *P*-Sasakian structure.

In §2, we recall some facts about hypersurfaces in a locally product Riemannian manifold and some properties about infinitesimal variations of hypersurfaces.

In §3, we prove the above theorem and state one corollary. Finally, we prove that there does not exist an infinitesimal projective normal variation satisfying the condition (2.30) of a *P*-Sasakian hypersurface in a locally product Riemannian manifold.

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Throughout this paper, we assume that manifolds are class C^∞ and orientable and the tensor fields are always class C^∞.

2. PRELIMINARIES

Let M^n be an n-dimensional locally product Riemannian manifold covered by a system of coordinate neighbourhood $\{U, x^\lambda\}$ and F_μ^λ and $g_{\mu\lambda}$ be respectively its almost product structure and its associated positive definite Riemannian metric, where and in the sequel the indices λ, μ, ν ... run over the range 1, 2, ..., n.

Then by definition we have

$$F_\mu^\gamma F_\gamma^\lambda = \delta_\mu^\lambda \qquad (F_\mu^\lambda \neq \delta_\mu^\lambda) \qquad \ldots(2.1)$$

$$g_{\epsilon\gamma} F_\mu^\epsilon F_\lambda^\gamma = g_{\mu\lambda} \qquad \ldots(2.2)$$

$$\nabla_\nu F_\mu^\lambda = 0 \qquad \ldots(2.3)$$

where the operator ∇_λ denotes the covariant differentiation with respect to $g_{\mu\lambda}$ (Tachibana 1960).

Let M^{n-1} be a hypersurface of M^n covered by a system of coordinate neighbourhood $\{V, y^i\}$ and g_{ji} be the induced metric tensor, where and in the sequel the indices $h, i, j, \ldots$ run over the range 1, 2, ..., $n-1$.

We put the local representation of M^{n-1} by

$$x^\lambda = x^\lambda(y^i) \qquad \ldots(2.4)$$

and put

$$B_i^\lambda = \partial_i x^\lambda \qquad (\partial_i = \partial/\partial y^i). \qquad \ldots(2.5)$$

Then B_i^λ are $n-1$ linearly independent vectors of M^n tangent to M^{n-1}. If we denote C^λ the unit normal to M^{n-1}, then we have

$$g_{\mu\lambda} B_j^\mu B_i^\lambda = g_{ji}, \qquad g_{\mu\lambda} C^\mu C^\lambda = 1, \qquad g_{\mu\lambda} B_j^\mu C^\lambda = 0 \qquad \ldots(2.6)$$

where we choose C^λ in such a way that C^λ, B_i^λ form the positive sense of M^n and that B_i^λ form the positive sense of M^{n-1}.

Now the transformation $F_\mu^\lambda B_j^\mu$ of B_j^μ by F_μ^λ can be written as

$$F_\mu^\lambda B_j^\mu = f_j^i B_i^\lambda + f_j C^\lambda \qquad \ldots(2.7)$$

where f^i_j and f_i are respectively a tensor field of type (1, 1) and a covariant vector field on M^{n-1}. And the transformation $F^\lambda_\mu C^\mu$ of C^μ by F^λ_μ can be written as

$$F^\lambda_\mu C^\mu = f^i B^\lambda_i + qC^\lambda \qquad \text{...(2.8)}$$

where $f^i = f_j g^{ji}$ and q is a scalar function on M^{n-1}.

Applying F^ν_λ to (2.7) and (2.8) and considering a tangential and a normal part, we respectively have

$$f^i_j f^i_l = \delta^i_j - f_j f^i, \qquad f^i_j f_i = -qf_j \qquad \text{...(2.9)}$$

$$f^j f^i_j = -qf^i, \qquad f_i f^i = 1 - q^2. \qquad \text{...(2.10)}$$

If the space M^{n-1} is *n-a*-invariant (Matsumoto 1979), then the function q is identically zero and the tensor fields (f^i_j, f_i, g_{ji}) are an almost paracontact metric structure (Satō 1976, 1977).

By virtue of the equations of Gauss and Weingarten, differentiating (2.7) and (2.8) along M^{n-1}, we respectively find

$$\nabla_k f^i_j = h_{kj} f^i + f_j h^i_k, \qquad \nabla_j f_i = qh_{ji} - f^l_i h_{lj} \qquad \text{...(2.11)}$$

$$\nabla_j f^i = qh^i_j - h^l_j f^i_l, \qquad \nabla_i q = -2f^l h_{li} \qquad \text{...(2.12)}$$

where the operator ∇_i denotes the van der Waerden-Bortolotti covariant differentiation, h_{ji} is the second fundamental tensor of M^{n-1} and $h^i_j = h_{jk} g^{ki}$.

Especially, if the space M^{n-1} is *n-a*-invariant and *P*-umbilical (Matsumoto 1979), that is, $q = 0$ and

$$h_{ji} = -g_{ji} + f_j f_i, \qquad \text{...(2.13)}$$

the almost paracontact metric structure satisfies

$$\nabla_j f_i = \nabla_i f_j, \qquad \nabla_j f^i = f^i_j \qquad \text{...(2.14)}$$

$$\nabla_k f^i_j = (-g_{kj} + f_k f_j) f^i + (-\delta^i_k + f_k f^i) f_j. \qquad \text{...(2.15)}$$

The above equations teach us the almost paracontact metric structure is a *P*-Sasakian structure (Satō 1976, 1977). We call such a hypersurface a *P*-Sasakian hypersurface in a locally product Riemannian manifold. Hereafter, we call this manifold a *P*-Sasakian hypersurface, for simplicity.

If a vector field u^i in an almost paracontact Riemannian manifold preserves the structure tensor invariant, that is, if the vector field u^i satisfies the relations

$$L(u)g_{ji} = 0, \qquad L(u)f^i_{\ j} = 0, \qquad L(u)f_i = 0 \qquad \text{...(2.16)}$$

then the vector field u^i is called an infinitesimal automorphism, where the operator $L(u)$ denotes the Lie differentiation with respect to u^i.

Now let an infinitesimal variation of a hypersurface M^{n-1} in a locally product Riemannian manifold M^n be given by

$$\bar{x}^\lambda = x^\lambda + v^\lambda(y)\,\epsilon \qquad \text{...(2.17)}$$

v^λ being a vector field on M^n defined along M^{n-1} that is called a variation vector of (2.17), where ϵ is an infinitesimal (Yano 1978). Then we have

$$\bar{B}^\lambda_i = B^\lambda_i + (\partial_i v^\lambda)\epsilon \qquad \text{...(2.18)}$$

where $\bar{B}^\lambda_i = \partial_i \bar{x}^\lambda$ are $n-1$ linearly independent vectors tangent to the varied hypersurface at the varied point $(\bar{x}^\lambda)$.

We displace the vector $\bar{B}^\lambda_i$ parallely from the varied point $(\bar{x}^\lambda)$ to the original point (x^λ) and put them $\tilde{B}^\lambda_i$.

Then $\tilde{B}^\lambda_i$ can be written as

$$\tilde{B}^\lambda_i = \bar{B}^\lambda_i + \{{}_\nu{}^\lambda{}_\mu\}\,(x + v\epsilon)\, v^\nu \bar{B}^\mu_i\, \epsilon \qquad \text{...(2.19)}$$

from which

$$\tilde{B}^\lambda_i = \bar{B}^\lambda_i + (\nabla_i v^\lambda)\,\epsilon \qquad \text{...(2.20)}$$

neglecting the terms of order higher than one with respect to ϵ, where $\{{}_\nu{}^\lambda{}_\mu\}$ are the Christoffel symbols with respect to $g_{\mu\lambda}$ and we define $\nabla_i v^\lambda$ as

$$\nabla_i v^\lambda = \partial_i v^\lambda + \{{}_\nu{}^\lambda{}_\mu\}\, B^\nu_i\, v^\mu. \qquad \text{...(2.21)}$$

In the sequel, we always neglect terms of order higher than one with respect to ϵ.

If we put

$$\delta B^\lambda_i = \tilde{B}^\lambda_i - B^\lambda_i \qquad \text{...(2.22)}$$

and

$$v^\lambda = v^i B^\lambda_i + \theta C^\lambda \qquad \text{...(2.23)}$$

we have

$$\delta B^\lambda_j = [(\nabla_j v^i - \theta h^i_{\ j})\, B^\lambda_i + (\theta_j + h_{ji} v^i)\, C^\lambda]\,\epsilon \qquad \text{...(2.24)}$$

where v^i and θ are a vector field and a scalar function on M^{n-1}, respectively, and $\theta_j = \partial_j\theta$. We call the δB_i^λ an infinitesimal variation of B_i^λ.

When $\theta = 0$, that is, when the variation vector v^λ is tangent to the hypersurface, we say that the variation is tangential. When $v^i = 0$, that is, when the variation vector v^λ is normal to the hypersurface we say that the variation is normal. When $v^\lambda = 0$, that is, when the variation vector is identically zero, we say that the variation is trivial.

We denote by $\bar{C}^\lambda$ the unit normal to the varied hypersurface. We displace $\bar{C}^\lambda$ parallely from the point $(\bar{x}^\lambda)$ to the point (x^λ) and call it $\tilde{C}^\lambda$. Then $\tilde{C}^\lambda$ can be written as

$$\tilde{C}^\lambda = \bar{C}^\lambda + \{{}_\nu{}^\lambda{}_\mu\}\,(x + v\epsilon)\;v^\nu \bar{C}^\mu\epsilon. \qquad \ldots(2.25)$$

If we put

$$\delta C^\lambda = \tilde{C}^\lambda - C^\lambda \qquad \ldots(2.26)$$

then we can obtain

$$\delta C^\lambda = -(\theta^i + h_j^i v^j)\;B_i^\lambda\,\epsilon \qquad \ldots(2.27)$$

where $\theta^i = \theta_j g^{ji}$.

Furthermore, we have the following formulas about infinitesimal variations of the induced geometric objects (Matsumoto to appear) :

$$\left.\begin{aligned}
\delta f_j^i &= [Lf_j^i - \theta(h_l^i f_j^l - f_l^i h_j^l) + \theta_j f^i + f_j\theta^i]\,\epsilon \\
\delta f_i &= [Lf_i - \theta f_l h_i^l + q\theta_i + f_i^l\theta_l]\,\epsilon \\
\delta f^i &= [Lf^i - f_l^i\theta^l + \theta f^l h_l^i + q\theta^i]\,\epsilon \\
\delta g_{ji} &= [Lg_{ji} - 2\theta h_{ji}]\,\epsilon, \quad \delta q = [Lq - 2f^i\theta_i]\,\epsilon
\end{aligned}\right\} \qquad \ldots(2.28)$$

where the operator L denotes the Lie differentiation with respect to the vector field v^i.

Especially if we assume that the manifold M^{n-1} is a P-Sasakian hypersurface and the variation preserves the function q invariant, then (2.28) can be written as

$$\left.\begin{aligned}
\delta f_j^i &= [Lf_j^i + \theta_j f^i + f_j\theta^i]\,\epsilon \\
\delta f_i &= [Lf_i + f_i^l\theta_l]\,\epsilon, \quad \delta f^i = [Lf^i - f_l^i\theta^l]\,\epsilon \\
\delta g_{ji} &= [Lg_{ji} + 2\theta(g_{ji} - f_j f_i)]\,\epsilon
\end{aligned}\right\} \qquad \ldots(2.29)$$

and

$$f^i\theta_i = 0. \qquad \ldots(2.30)$$

3. Infinitesimal Projective Variations

Let M^{n-1} be a hypersurface of a locally product Riemannian manifold M^n.

If an infinitesimal variation (2.17) of M^{n-1} satisfies the relation

$$\delta \left\{ {}_j{}^h{}_i \right\} = (\delta_j^h p_i + \delta_i^h p_j)\,\epsilon \qquad \ldots(3.1)$$

then we call such a variation an infinitesimal projective variation. An infinitesimal projective variation satisfying $p_i = 0$ identically is called an infinitesimal affine variation, where p_i is a certain vector field on M^{n-1} and $\left\{ {j}^{\;h}{}_{i} \right\}$ are the Christoffel symbols with respect to g_{ji} (Yano 1978).

In this section, we assume that the manifold M^{n-1} is always a P-Sasakian hypersurface satisfying $f^2 - (n-2)^2 \neq 0\,(f = f_i^i = \text{trace}\,(f_j^i))$ and an infinitesimal variation (2.17) of M^{n-1} preserves the function q that is, the variation satisfies (2.29) and (2.30).

Remark : The condition $f^2 - (n-2)^2 \neq 0$ teaches us the manifold M^{n-1} is not SP-Sasakian (Adati and Miyazawa 1977).

Now let us consider an infinitesimal projective variation of M^{n-1}. Then (3.1) holds.

On the other hand, we know (Yano 1978)

$$\delta \left\{ {k}^{\;i}{}_{j} \right\} = [L \left\{ {k}^{\;i}{}_{j} \right\} - \nabla_k(h_j^i\,\theta) - \nabla_j(h_k^i\,\theta) + \nabla^i(h_{kj}\theta)]\,\epsilon. \qquad \ldots(3.2)$$

Hence we have from (3.1) and (3.2)

$$L \left\{ {k}^{\;i}{}_{j} \right\} = \nabla_k(h_j^i\,\theta) + \nabla_j(h_k^i\,\theta) - \nabla^i(h_{kj}\theta) + \delta_k^i p_j + \delta_j^i p_k. \qquad \ldots(3.3)$$

Substituting (2.13) into (3.3), we find

$$L \left\{ {k}^{\;i}{}_{j} \right\} = 2\theta f_{kj} f^i + (-\delta_k^i + f_k f^i)\,\theta_j + (-\delta_j^i + f_j f^i)\,\theta_k$$

$$- (-g_{kj} + f_k f_j)\,\theta^i + \delta_k^i p_j + \delta_j^i p_k. \qquad \ldots(3.4)$$

By virtue of (3.4) and the formula (Yano 1965)

$$LR_{kji}^h = \nabla_k L \left\{ {j}^{\;h}{}_{i} \right\} - \nabla_j L \left\{ {k}^{\;h}{}_{i} \right\}$$

we obtain

$$LR_{kji}^h = (\theta_k f_{ji} - \theta_j f_{ki}) f^h + (f_k^h\,\theta_j - f_j^h\,\theta_k) f_i + 2\theta\,\{(g_{ji} f_k$$

$$- g_{ki} f_j) f^h + f_{ji} f_k^h - f_{ki} f_j^h\} + (-\delta_j^h + f_j f^h)\,\nabla_k \theta_i -$$

(*equation continued on p.* 117)

$$- (-\delta_k^h + f_k f^h) \nabla_j \theta_i - (-g_{ji} + f_j f_i) \nabla_k \theta^h + (-g_{ki} + f_k f_i) \nabla_j \theta^h$$

$$+ (f_j f_k^h - f_k f_j^h) \theta_i - (f_j f_{ki} - f_k f_{ji}) \theta^h + \delta_j^h \nabla_k p_i - \delta_k^h \nabla_j p_i$$

$$+ \delta_i^h (\nabla_k p_j - \nabla_j p_k) \qquad \ldots(3.5)$$

where R_{kji}^h denotes the curvature tensor with respect to g_{ji}.

Summing up (3.5) with i and h, we have

$$\nabla_k p_j - \nabla_j p_k = 0 \qquad \ldots(3.6)$$

that is, the vector field p_i is closed. Hence (3.5) can be written as

$$LR_{kji}^h = (\theta_k f_{ji} - \theta_j f_{ki}) f^h + (f_k^h \theta_j - f_j^h \theta_k) f_i$$

$$+ 2\theta \{(g_{ji} f_k - g_{ki} f_j) f^h + f_{ji} f_k^h - f_{ki} f_j^h\} + (f_j f_k^h - f_k f_j^h) \theta_i$$

$$- (f_j f_{ki} - f_k f_{ji}) \theta^h + (-\delta_j^h + f_j f^h) \nabla_k \theta_i - (-\delta_k^h + f_k f^h) \nabla_j \theta_i$$

$$- (-g_{ji} + f_j f_i) \nabla_k \theta^h + (-g_{ki} + f_k f_i) \nabla_j \theta^h + \delta_j^h \nabla_k p_i$$

$$- \delta_k^h \nabla_j p_i. \qquad \ldots(3.7)$$

Summing up (3.7) with k and h, we get

$$LR_{ji} = 2\theta f f_{ji} + f(f_j \theta_i + f_i \theta_j) + (n-4) \nabla_j \theta_i - 2(f_j f_i^k + f_i f_j^k) \theta_k$$

$$+ (\nabla_k \theta^k)(g_{ji} - f_j f_i) - (n-2) \nabla_j p_i \qquad \ldots(3.8)$$

where R_{ji} denotes the Ricci tensor with respect to g_{ji}.

Transvecting (3.8) with f^i and taking account of (2.30) and the formula (Satō and Matsumoto 1979)

$$R_{ji} f^i = -(n-2) f_j$$

we have

$$-(n-2) Lf_j - R_{ji} Lf^i = f\theta_j + (n-2) f^i \nabla_j \theta_i - (n-2) f^i \nabla_j p_i. \qquad \ldots(3.9)$$

From which, we have

$$2f^j Lf_j = f^i f^j \nabla_j p_i. \qquad \ldots(3.10)$$

Now, transvecting (3.7) with $f_h f^k$ and taking account of the formula (Satō and Matsumoto 1979)

$$R^h_{kji} f_h = g_{ki} f_j - g_{ji} f_k$$

we obtain

$$f^k(Lg_{ki}) f_j - Lg_{ji} = 2\theta(g_{ji} - f_j f_i) + f_j f^k \nabla_k p_i - \nabla_j p_i. \qquad ...(3.11)$$

In general, since we know the relation

$$f^k Lg_{ki} = Lf_i - g_{ki} Lf^k$$

(3.11) can be written as

$$(Lf_i - g_{ki} Lf^k) f_j - Lg_{ji} = 2\theta(g_{ji} - f_j f_i) + f_j f^k \nabla_k p_i - \nabla_j p_i. \qquad ...(3.12)$$

Transvecting (3.12) with f^i and taking account of (3.10), we find

$$f^i Lg_{ji} = f^i \nabla_j p_i. \qquad ...(3.13)$$

Substituting (3.13) into (3.11) and using (3.6), we get

$$Lg_{ji} = -2\theta\,(g_{ji} - f_j f_i) + \nabla_j p_i. \qquad ...(3.14)$$

By virtue of (3.14) and the formula (Yano 1965)

$$L \left\{ {h \atop k\ \ j} \right\} = \tfrac{1}{2} g^{hi} (\nabla_k Lg_{ji} + \nabla_j Lg_{ki} - \nabla_i Lg_{kj})$$

we obtain

$$L \left\{ {h \atop k\ \ j} \right\} = -\theta_k(\delta^h_j - f_j f^h) - \theta_j(\delta^h_k - f_k f^h) + \theta^h(g_{kj} - f_k f_j)$$
$$+ 2\theta f_{kj} f^h + \tfrac{1}{2} (\nabla_j \nabla_k p_i - R^l_{kji} p_l)\, g^{ih}. \qquad ...(3.15)$$

Comparing with (3.4) and (3.15), we have

$$\nabla_j \nabla_i p^h + R^h_{lji} p^l = 2(\delta^h_j p_i + \delta^h_i p_j). \qquad ...(3.16)$$

Thus we have from (3.6) and (3.16)

Proposition 1 — For an infinitesimal projective variation (3.1) satisfying (2.30) of a *P*-Sasakian hypersurface, the vector field p_i is a closed projective Killing vector field with an associated vector field $2p_i$.

Remark : In Proposition 1, we do not use the condition $f^2 - (n-2)^2 \neq 0$.

On the other hand, Ogata (1978) has proved the following:

Proposition 2 — In a *P*-Sasakian manifold with $f^2 - (n-2)^2 \neq 0$, each projective Killing vector field is an infinitesimal automorphism.

By virtue of (3.1), (3.14) and Propositions 1 and 2, we have:

Proposition 3 — An infinitesimal projective variation (3.1) satisfying (2.30) of a *P*-Sasakian hypersurface is an infinitesimal affine one. Furthermore, since the vector field p_i is identically zero, then the vector field v^i is an f-conformal Killing vector field (Matsumoto 1977).

For an f-conformal Killing vector field, we have proved the following (Matsumoto 1977) :

Proposition 4 — In a compact orientable P-Sasakian manifold, an f-conformal Killing vector field is an infinitesimal automorphism.

Let the hypersurface M^{n-1} be a compact P-Sasakian hypersurface, then from (3.14) and the above proposition we have $\theta = 0$, i.e., our variation is tangential. Furthermore, by virtue of the above result and (2.29), we can show that the variation preserves P-Sasakian structure invariant. We know that a compact P-Sasakian hypersurface satisfies $f^2 - (n-2)^2 \neq 0$ (Sasaki 1981). Thus our main theorem was proved.

As the special case of the main theorem, we have

Corollary — For an infinitesimal affine variation satisfying (2.30) of a P-Sasakian hypersurface M^{n-1}, the vector field v^i is an f-conformal Killing vector field. If the manifold M^{n-1} is compact then the variation is tangential and preserves the P-Sasakian structure.

Finally, we shall consider an infinitesimal projective normal variation of a P-Sasakian hypersurface. Then (3.4) can be written as

$$2\theta f_{ji} f^h + (-\delta_i^h + f_i f^h)\,\theta_j + (-\delta_j^h + f_j f^h)\,\theta_i$$

$$- (-g_{ji} + f_j f_i)\,\theta^h + \delta_j^h p_i + \delta_i^h p_j = 0. \qquad \ldots(3.17)$$

Transvecting (3.17) with f_h, we have

$$2\theta f_{ji} + f_j p_i + f_i p_j = 0 \qquad \ldots(3.18)$$

from which

$$p_j = -(f^i p_i)\, f_j. \qquad \ldots(3.19)$$

Transvecting (3.17) with g^{ji}, we obtain

$$2(\theta f - p^i f_i)\, f^h - (n-4)\,\theta^h = 0. \qquad \ldots(3.20)$$

By virtue of (2.30) and the above equation, we find

$$\theta f = f^i p_i. \qquad \ldots(3.21)$$

Substituting (3.19) and (3.21) into (3.18), we get

$$\theta(f_{ji} - f f_j f_i) = 0$$

from which, we get $\theta = 0$, that is, the infinitesimal variation is trivial. Thus we have

Theorem 5 — Except for a trivial infinitesimal variation, there does not exist an infinitesimal projective normal variation satisfying (2.30) of a P-Sasakian hypersurface in a locally product Riemannian manifold.

Remark : In the above theorem, we does not use the condition $f^2 - (n-2)^2 \neq 0$.

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