Indian J. pure appl. Math., 13(2) : 121-126, February 1982

CLASS OF LINE COMPLEXES IN THE SECOND DIFFERENTIAL NEIGHBOURHOOD OF THE RAY IN THE FLAG SPACE F_3

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(*Received* 10 *October* 1980)

The present article is devoted to the differential geometry of line complexes whose the centres of the ray are the centres of inflection in the flag space F_3. For this class of line complexes, existence theorem is proved and its integral-free representation is given. The method adopted here based on Cartan's differential calculus (Finikov 1948).

1. INTRODUCTION

The flag space F_3 is regarded as a 3-dimensional projective space P_3 contains degenerate absolutum consists of a plane with an invariant line in it and an invariant point on this line. The geometry under consideration, which is the differential geometry of a sixfold group in 3-dimensional space, shows much resemblance with the Euclidean geometry. In P_3 we use x^1, x^2, x^3, x^4 as homogeneous point coordinates. We consider the geometry belonging to the following sixfold group of projective transformations:

$$\begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \\ \bar{x}^4 \end{bmatrix} = \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix}$$

Our geometry has the properties of the affine and therefore also of the projective geometry. As usual in affine geometry, we call the invariant plane $x^4 = 0$ the ideal plane and every configuration in this plane will be called ideal. For the points which do not lie in the ideal plane, we take $x^4 = 1$.

The distance of the ordered pair of points $P(x^1, x^2, x^3)$ and $Q(y^1, y^2, y^3)$ is $y^3 - x^3$. If this distance is zero for two different points, we define $y^2 - x^2$ as distance of second order and if this distance is zero too, we define $y^1 - x^1$ as distance of third order. The angle between the ordered pair of planes $x^1 + \xi_2 x^2 + \xi_3 x^3 + \xi_4 x^4 = 0$ and $x^1 + \eta_2 x^2 + \eta_3 x^3 + \eta_4 x^4 = 0$ is given by $\xi_2 - \eta_2$. If this angle is zero, we may introduce an angle of higher order as in the distance (Rosenfeld 1969).

We choose a moving frame conjugate to any arbitrary manifold immersed in the flag space F_3 as a coordinate tetrahedron $T(\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, where the vertices $\bar{A}_1$, $\bar{A}_2$,

$\bar{A}_3$ are the points at infinity, $\bar{A}_4$ is a proper point such that $\bar{A}_4\bar{A}_1$, $\bar{A}_4\bar{A}_2$, $\bar{A}_4\bar{A}_3$ form orthogonal triad, the invariant point is $\bar{A}_1$ and the invariant line $\bar{A}_1\bar{A}_2$ lie on the invariant plane $\bar{A}_1\bar{A}_2\bar{A}_3$. The absolutum in the mentioned moving frame T consists of the plane $x^4 = 0$, the line $x^3 = 0$, $x^4 = 0$ and the point (1,0, 0, 0). The geometry is completely self-dual (Redie 1968).

The fundamental equations of the moving frame T are

$$d\begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \bar{A}_3 \\ \bar{A}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \omega_2^1 & 0 & 0 & 0 \\ \omega_3^1 & \omega_3^2 & 0 & 0 \\ \omega_4^1 & \omega_4^2 & \omega_4^3 & 0 \end{bmatrix} \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \bar{A}_3 \\ \bar{A}_4 \end{bmatrix}, \qquad \ldots(1)$$

and the structural equations (the integrability conditions) are

$$D\begin{bmatrix} \omega_2^1 \\ \omega_3^1 \\ \omega_3^2 \\ \omega_4^1 \\ \omega_4^2 \\ \omega_4^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \omega_3^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \omega_4^2 & \omega_4^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_4^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Lambda \begin{bmatrix} \omega_2^1 \\ \omega_3^1 \\ \omega_3^2 \\ \omega_4^1 \\ \omega_4^2 \\ \omega_4^3 \end{bmatrix}, \qquad \ldots(2)$$

where ω_i^j are Pfaff's differential form, D denotes the exterior differentiation operator and Λ the exterior product between the differential forms.

2. Arbitrary Line Complex Related to a Canonical Frame in F_3

Let, in F_3, u, v, w be a system of curvilinear coordinates; the admissible variables (u, v, w) are taken from an open neighbourhood of C^3. Consider a line complex immersed in F_3, which generated by the ray $l = (\bar{A}_3\bar{A}_4)$, $l = l(u, v, w)$, we denote by $(\bar{A}_3\bar{A}_4)$ the Grassman coordinates of the line l. A coordinate tetrahedron $T(\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$ with the coordinate planes $(\bar{A}_3\bar{A}_4\bar{A}_2)$ and $(\bar{A}_3\bar{A}_4\bar{A}_1)$ coincident with the planes corresponding to the vertices $\bar{A}_3$ and $\bar{A}_4$ respectively in the normal correlation of the ray l is called a canonical tetrahedron and the vertices $\bar{A}_3$ and $\bar{A}_4$ are called the centres of the ray l of line complex (Kovansov 1963). The differential equation of this complex related to its canonical frame in the first differential neighbourhood of the ray l has the form

$$\omega_4^1 = k\,\omega_3^2 \qquad \ldots(3)$$

where k is an invariant of the first differential neighbourhood and is called the curvature of the line complex. Excludes from our consideration the holonomic complex $\omega_3^2 = 0$. Exterior differentiation of eqn. (3) and using Cartan's lemma we have

$$\begin{bmatrix} \omega_2^1 \\ -\omega_4^3 \\ dk \end{bmatrix} = \begin{bmatrix} P & \alpha & \beta \\ \alpha & q & \gamma \\ \beta & \gamma & r \end{bmatrix} \begin{bmatrix} \omega_4^2 \\ \omega_3^1 \\ \omega_3^2 \end{bmatrix}, \qquad \ldots(4)$$

where p, α, β, q, γ, r, are the invariants of the second differential neighbourhood of the ray l and the forms ω_4^2, ω_3^1 and ω_3^2 are basis forms which is linear combination of the differentials du, dv and dw.

Lemma 1 — Any arbitrary line complex immersed in the flag space F_3 can be stratified into one-parameter family of holonomic line congruences.

PROOF : For the line complex (4), the differential equation

$$\omega_3^2 = 0 \qquad \ldots(5)$$

automatically satisfies $D\omega_3^2 \equiv 0$, then eqn. (5) determines a holonomic line congruence belonging to the arbitrary line complex (4). It follows that, this complex can be distributed into one-parameter family of line congruences (5).

To find the focal points of the line congruence (5), let $\bar{F} = \bar{A}_3 + t\bar{A}_4$ be any analytic point on the ray $l = (\bar{A}_3\bar{A}_4)$. This point will be a focal point if $d\bar{F} \equiv 0$ (mod $\bar{F}$), which gives the equations

$$\omega_3^1 + t\omega_4^1 = 0,\ \omega_3^2 + t\omega_4^2 = 0,\ (\omega_4^1 = 0 = \omega_3^2).$$

This means that, the focal points corresponding to $t = 0$, $t = \infty$ are the centres $\bar{A}_3$, $\bar{A}_4$ of the ray l respectively. From the differentials

$$d\bar{A}_3 = \omega_3^1\, \bar{A}_1,\ d(\bar{A}_3\bar{A}_1) = 0,\ d\bar{A}_4 = \omega_4^2\, (\bar{A}_2 - \alpha\bar{A}_3) - q\omega_3^1\, \bar{A}_3,$$

we see that, the focal surfaces described by the focal points $\bar{A}_3$, $\bar{A}_4$ are the fixed line $(\bar{A}_3\bar{A}_1)$, arbitrary surface σ with tangent plane $(\bar{A}_3\bar{A}_4\bar{A}_2)$ respectively. The tangent plane to the focal surface σ coincident with the plane corresponding to the centre $\bar{A}_3$ of the ray l in the normal correlation.

3. EXISTENCE THEOREM

The centres of inflection of the ray of a given complex are defined in Kovansov (1963) as these points having the property that, their corresponding line cone posses this ray as a singular generator. The point $\bar{M} = \bar{A}_3 + t\bar{A}_4$ will be a centre of inflection of the ray l if its abscissa satisfies the equation

$$qk^2t^4 - 2k\gamma t^3 + (2k\alpha + r)\, t^2 - 2\beta t + p = 0. \qquad \ldots(6)$$

This means that, there are four centres of inflection on the ray l of the line complex (4).

If the centres $\bar{A}_3(t = 0)$ and $\bar{A}_4(t = \infty)$ of the ray l of the complex (4) are centres of inflection, then $t = 0$ and $t = \infty$ must satisfy eqn. (6). Thus we have

$p = q = 0$, and the differential equations of a line complex for which centres of the ray l are centres of inflection are

$$\begin{bmatrix} \omega_4^1 \\ \omega_2^1 \\ -\omega_4^3 \\ dk \end{bmatrix} = \begin{bmatrix} 0 & 0 & k & 0 \\ 0 & \alpha & \beta & 0 \\ \alpha & 0 & \gamma & 0 \\ \beta & \gamma & r & 0 \end{bmatrix} \begin{bmatrix} \omega_4^2 \\ \omega_3^1 \\ \omega_3^2 \\ 0 \end{bmatrix}. \quad \ldots(7)$$

Exterior differentiation of the last three equations of (7) and using Cartan's lemma, we get

$$d\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ r \end{bmatrix} = \begin{bmatrix} 0 & 0 & \lambda_1 - \alpha^2 & 0 \\ 0 & \lambda_1 & \lambda_2 & 0 \\ \lambda_1 & 0 & \lambda_3 & 0 \\ \lambda_2 + \alpha\beta & \lambda_3 + \alpha\gamma & \lambda_4 & 0 \end{bmatrix} \begin{bmatrix} \omega_4^2 \\ \omega_3^1 \\ \omega_3^2 \\ 0 \end{bmatrix}. \quad \ldots(8)$$

From (7) and (8) it follows that, the number of independent parameter $N = 4$ $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Using Cartan's common method, we have the number of independent characteristic forms $\bar{q} = 4(d\alpha, d\beta, d\gamma, dr)$, the number of independent quadratic exterior forms $S_1 = 3$, and the equality $\bar{q} = S_1 + S_2$, gives that $S_2 = 1$. The Cartan's number $Q = S_1 + 2S_2 = 5$ is greater than the number of independent parameters. This means that, the system of differential equations (7) is not in involution.

For this reason we consider the system of differential equations. (8). Exterior differentiation of (8) and using Cartan's lemma, we obtain

$$d\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \mu_1 & 0 \\ 0 & \mu_1 + \alpha\lambda_1 & \mu_2 & 0 \\ \mu_1 + \alpha\lambda_1 & 0 & \mu_3 & 0 \\ \mu_2 + \beta\lambda_1 + 2\alpha\lambda_2 & \mu_3 + 2\alpha\lambda_3 + \gamma\lambda_1 & \mu_4 & 0 \end{bmatrix} \begin{bmatrix} \omega_4^2 \\ \omega_3^1 \\ \omega_3^2 \\ 0 \end{bmatrix} \quad \ldots(9)$$

From (9) it follows that the number of independent parameter $N = 4(\mu_1, \mu_2, \mu_3, \mu_4)$, $\bar{q} = 4(d\lambda_1, d\lambda_2, d\lambda_3, d\lambda_4)$, $S_1 = 4$ and the equality $\bar{q} = S_1 + S_2$, gives $S_2 = 0$ (the oldest character). The Cartan's number $Q = S_1 = 4 = N$ the number of independent parameters. This means that, the system of differential equations. (7) is in involution andexists within four functions of one argument. Hence we have the following existence theorem.

Theorem 1 — The line complex for which the centres of the ray are centres of inflection exists within four arbitrary functions of one argument.

4. The Geometrical Construction

The following theorems give the integral-free representation of the line complex (7).

Theorem 2 — If the ray $l = (\bar{A}_3\bar{A}_4)$ generates the line complex (7), then the centre of the ray $\bar{A}_4$ describes a ruled surface its generator is $m = (\bar{A}_4, \bar{A}_2 - \alpha\bar{A}_3)$.

PROOF : From the differentials

$$d\bar{A}_4 = \omega_3^2 \{k\bar{A}_1 - \gamma \bar{A}_3\} + \omega_4^2 \{\bar{A}_2 - \alpha\bar{A}_3\},$$

$$dm \equiv \omega_3^2 \{k(\bar{A}_1, \bar{A}_2 - \alpha\bar{A}_3) + (\bar{A}_4, \beta\bar{A}_1 - \alpha\bar{A}_2 - (\lambda_1 - \alpha^2)\bar{A}_3)\},$$

it follows that, $d\bar{A}_4$ depends on two principal forms ω_3^2, ω_4^2 and dm depends on one principal form ω_3^2, then the centre of the ray $\bar{A}_4$ describes a ruled surface Σ with generator m. The ruled surface Σ has tangent plane Σ with tangential coordinates.

$$\sigma \equiv (\bar{A}_4\,\bar{A}_2\,\bar{A}_1) - \alpha(\bar{A}_4\,\bar{A}_3\,\bar{A}_1) + (\gamma/k)\,(\bar{A}_4\,\bar{A}_3\,\bar{A}_2).$$

From the above theorem, we have the geometrical meaning of the invariants α γ in the second differential neighbourhood, the invariant $-\alpha$ equal to the angle between σ and the coordinate plane $x^3 = 0$ and the invariant γ/k equal to the angle of second order between σ and the coordinate plane $x^3 = 0$.

Lemma 2 — The partial differential equation $\omega_4^1 = 0$ determines a hyperbolic linear line congruence with two directrices.

PROOF : For the line complex (7), the equation $\omega_4^1 = 0$ satisfies $D\omega_4^1 \equiv 0$ (mod ω_4^1), i.e., $\omega_4^1 = 0$ determines a holonomic congruence belonging to the line complex (7). The focal points of this line congruence are the centres of the ray $l = (\bar{A}_3\bar{A}_4)$. From the differentials

$$d\bar{A}_3 = \omega_3^1\bar{A}_1,\ d\bar{A}_4 = \omega_4^2\{\bar{A}_2 - \alpha\bar{A}_3\},\ (\omega_3^2 = 0 = \omega_4^1)$$

and $d(\bar{A}_3\bar{A}_1) = 0$, $dm = d(\bar{A}_4, \bar{A}_2 - \alpha\bar{A}_3) = 0$, it follows that, the focal points $\bar{A}_3$ and $\bar{A}_4$ describes the fixed lines $(\bar{A}_3\bar{A}_1)$ and m respectively. Hence the line congruence $\omega_4^1 = 0$, is a hyperbolic linear congruence with m and $(\bar{A}_3\bar{A}_1)$ as the two directrices.

Corollary — The line complex (7) admitting a fibration into one-parameter family of a hyperbolic linear line congruence.

From

$$d\{\bar{A}_3 + t_1\bar{A}_1\} = \{\omega_3^1 + dt_1\}\,\bar{A}_1,\ D\{\omega_3^1 + dt_1\} \equiv 0 \ (\text{mod } \omega_4^1),$$

$$d\{\bar{A}_4 + t_2(\bar{A}_2 - \alpha\bar{A}_3)\} = \{\omega_4^2 + dt_2\}\{\bar{A}_2 - \alpha\bar{A}_3\},\ D\{\omega_4^2 + dt_2\}$$
$$\equiv 0 \ (\text{mod } \omega_4^1),$$

it follows that, $t_1 = f(\tau)$, $t_2 = g(\tau)$ and any point on the directrix $(\bar{A}_3\bar{A}_1)$ of the linear congruence $\omega_4^1 = 0$, determines some point on the second directrix $m = (\bar{A}_4, \bar{A}_2 - \alpha\bar{A}_3)$. This means that any point on the line $(\bar{A}_3\bar{A}_1)$ depends on the parameter τ of the ruled surface generated by m (i.e., depends on one arbitrary function of one parameter).

The following theorem gives the integral-free representation of the line complex (7).

Theorem 3 — We consider a ruled surface. Construct a hyperbolic linear congruence, for which the generator of the ruled surface and a line on the ideal plane are directrices. All these congruences constructs the line complex under discussion.

PROOF : We choose a frame of reference in the form of a coordinate canonical tetrahedron $T(\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, such that the vertices $\bar{A}_1$ and $\bar{A}_3$ are placed on a generator of the ruled surface, this generator intersects the ideal plane at a point $\bar{P}$. On the line $(\bar{A}_3\bar{P})$ we place the point $\bar{A}_2$, in this case $\bar{P} = \bar{A}_2 - \alpha\bar{A}_3$. The line $(\bar{A}_3\bar{A}_1)$ and $(\bar{A}_4\ \bar{P})$ will be the directrices of a hyperbolic linear congruence.

The line $(\bar{A}_1\bar{A}_3)$ is a directrix of a linear congruence generated by $l = (\bar{A}_3\bar{A}_4)$ if and only if $\omega_3^2 = 0$ and $\bar{A}_4$ describes the generator $m = (\bar{A}_4, \bar{A}_2 - \alpha\bar{A}_3)$ if and only if $\omega_4^1 = 0$. If $l = (\bar{A}_3\bar{A}_4)$ coincident with the ray of the line complex, the principal forms of the moving frame will be ω_4^1, ω_3^2, ω_3^1 and ω_4^2. Since $\omega_4^1 = 0$ if and only if $\omega_3^2 = 0$, then the forms ω_4^1 and ω_3^2 must be related by

$$\omega_4^1 = k\omega_3^2. \qquad \text{...(10)}$$

Exterior differentiation of (10) leads to (4). The line m is a generator of the ruled surface and a directrix of linear congruence $\omega_3^2 = 0$ if and only if

$$dm \equiv \omega_3^2(k\bar{A}_1 - \gamma\bar{A}_3, \bar{A}_2 - \alpha\bar{A}_3) + (\bar{A}_4, \{\omega_2^1 - \alpha\omega_3^1\}\bar{A}_1 - \alpha\omega_3^2\bar{A}_2 - d\alpha\bar{A}_3)$$

depends on the principal form ω_3^2, this leads to

$$d\alpha\Lambda\,\omega_3^2 = 0,\ \{\omega_2^1 - \alpha\omega_3^1\}\Lambda\omega_3^2 = 0. \qquad \text{...(11)}$$

From (4) using eqns. (11), we have

$$q = p = 0. \qquad \text{...(12)}$$

The relation (12) characterize the constructed line complex (7).

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