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MULTI-SCALAR SYSTEMS AND DISTRIBUTION MATRICES

HANSRAJ GUPTA*

Panjab University, Chandigarh 160 014

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In this paper, we define the degree and weight of numbers represented in a multi-scalar system and study how the numbers are distributed with regard to their degrees and weights.

1. INTRODUCTION

Closely related with the problem of finding, in the Hermite normal form, the number of right divisors of a diagonal matrix the elements of which are powers of an arbitrary prime p, is a system of representation of nonnegative integers in what we call here a multi-scalar system. In this paper, we define the degree and weight of numbers so represented and study how the numbers are distributed with regard to their degrees and weights.

2. THE MULTI-SCALAR SYSTEMS

Let $D_k = (d_1, d_2, \ldots, d_k)$ be a sequence of natural numbers such that

$$1 \leqslant d_1 \leqslant d_2 \leqslant \ldots \leqslant d_k. \qquad \ldots(2.1)$$

Then, we assert that every nonnegative integer which is

$$< (d_1 + 1)(d_2 + 1) \ldots (d_k + 1) \qquad \ldots(2.2)$$

can be represented in the form

$$a_1 a_2 \ldots a_k \qquad \ldots(2.3)$$

where

$$0 \leqslant a_j \leqslant d_j, \quad j \leqslant k.$$

The multiscale, in this case, is given by

$$(d_1 + 1,\ d_2 + 1, \ldots, d_k + 1). \qquad \ldots(2.4)$$

It will be convenient to speak of this multi-scale as the multi-scale generated by D_k.

The representation for any n satisfying (2.2) is obtained as follows:

*Mailing address : 402 Mumfordganj, Allahabad 211002.

Write $n = (d_k + 1) q_k + a_k,\ 0 \leqslant a_k \leqslant d_k;$

$$q_k = (d_{k-1} + 1) q_{k-1} + a_{k-1},\ 0 \leqslant a_{k-1} \leqslant d_{k-1};$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$q_j = (d_{j-1} + 1) q_{j-1} + a_{j-1},\ 0 \leqslant a_{j-1} \leqslant d_{j-1};$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$q_3 = (d_2 + 1) q_2 + a_2,\ 0 \leqslant a_2 \leqslant d_2.$$

Since

$$n < (d_1 + 1)(d_2 + 1) \ldots (d_k + 1),$$

we must have $0 \leqslant q_2 \leqslant d_1$.

Writing a_1 for q_2, the required representation for n is

$$a_1 a_2 \ldots a_{k-1} a_k. \qquad \ldots(2.5)$$

The a's in this representation will be spoken of as the digits or the elements in the multi-scalar representation of n. (2.5) may be considered a number of k digits.

The process for finding the representation for a given n can best be presented in the following form:

Let $d_1 = 3,\ d_2 = 4,\ d_3 = 6,\ d_4 = 7$. Then every number < 1120 is representable in the multi-scale (4, 5, 7, 8). For example 987 is representable and the representation can be found as follows:

8) 987
7) 123, remainder 3;
5) 17, remainder 4;
3, remainder 2.

Hence the required representation for 987 is 3243.

In a given multi-scale such a representation for a given n is unique. It is also easy to show that if the expression

$$a_1 a_2 \ldots a_k$$

represents a number in the multi-scale generated by a certain D_k then it must be

$$a_k + a_{k-1}(d_k + 1) + a_{k-2}(d_{k-1} + 1)(d_k + 1) + \ldots$$
$$+ a_1(d_2 + 1)(d_3 + 1) \ldots (d_k + 1). \qquad \ldots(2.6)$$

In what follows except for the numbers under study, all other numbers will be written in the decimal scale.

3. Complement, Degree and Weight of a Number

If $a_1 a_2 \ldots a_k$ is the representation of a number in the multi-scale generated by D_k, then

$$b_1 b_2 \ldots b_k \qquad \ldots(3.1)$$

is said to be the complement of $a_1 a_2 \ldots a_k$ if and only if

$$a_j + b_j = d_j, \quad \text{for each } j \leqslant k. \qquad \ldots(3.2)$$

Evidently, if B is the complement of A then A is the complement of B. We shall also speak of b_j as the complement of a_j and vice versa if

$$a_j + b_j = d_j.$$

In what follows, we shall use the letters a_j and b_j in this sense, unless stated otherwise.

Example — In the multi-scalar system generated by

$$d_1 = 3, \quad d_2 = 7, \quad d_3 = 8, \quad d_4 = 9 = d_5 :$$

the numbers 25046 and 12853 are complementary.

The degree of the element a_j in the number

$$a_1 a_2 \ldots a_j \ldots a_k \qquad \ldots(3\ 3)$$

is given by the sum

$$\sum_{t=1}^{j-1} \min(b_j, a_t), \quad \text{for } j > 1; \qquad \ldots(3.4)$$

and the degree of a_1 itself is taken to be zero.

The degree of (3.3) is the sum of the degrees of its elements.

For example, let us determine the degree of 25046 in the foregoing case.

Here the degree of 6 = min (3,2) + min (3,5) + min (3,0) + min (3,4)

= 2 + 3 + 0 + 3 = 8;

the degree of 4 = min (5,2) + min (5,5) + min (5,0) = 7;

the degree of 0 = min (8,2) + min (8,5) = 7;

the degree of 5 = min (2,2) = 2;

and the degree of 2 = 0.

Hence the degree of 25046 = 24.

Finally, the weight of a number written in a given multiscale is the sum of its elements or digits. Thus the weight of 25046 is 17.

Evidently, the weight of no number written in the multi-scale generated by D_k can exceed the sum

$$d_1 + d_2 + \ldots + d_k. \qquad \ldots(3.5)$$

We can now make the following statement:

The degree of a_j in (3.3) is the weight of the number obtained by replacing in the number

$$a_1 a_2 \ldots a_{j-1}$$

each $a_t > b_j$ by b_j.

We express this operation by writing

$$\text{degree of } a_j = W(b_j)\,(a_1 a_2 \ldots a_{j-1}). \qquad \ldots(3.6)$$

Example — For $d_1 = d_2 = d_3 = d_4 = 7$, the degree of 5 in 4315 $= W(2)\,(431) = 2 + 2 + 1 = 5$; while the degree of 1 therein $= W(6)\,(43) = 4 + 3 = 7$. (3.6) is of immense use in practice.

4. The Distribution Matrix

Given a multi-scale, we first find the degree and the weight of each number representable in that multi-scale. A number which is of degree u and weight w is said to be of the type (u, w). The number of numbers which are of the type (u, w) is denoted by $N(u, w)$. We now present in the form of a matrix the values of $N(u, w)$ for different values of (u, w). Such a matrix we call a distribution matrix.

To clarify ideas, we consider an example in detail.

Example — Let $d_1 = d_2 = 1$, $d_3 = 2$, $d_4 = 3$.

Then we have the following table:

Number	Type	Number	Type	Number	Type
0000	(0, 0)	0110	(3, 2)	1020	(4, 3)
0001	(0, 1)	0111	(3, 3)	1021	(4, 4)
0002	(0, 2)	0112	(3, 4)	1022	(3, 5)
0003	(0, 3)	0113	(1, 5)	1023	(1, 6)
0010	(1, 1)	0120	(3, 3)	1100	(4, 2)
0011	(1, 2)	0121	(3, 4)	1101	(4, 3)
0012	(1, 3)	0122	(2, 5)	1102	(4, 4)
0013	(0, 4)	0123	(0, 6)	1103	(2, 5)
0020	(2, 2)	1000	(3, 1)	1110	(5, 3)
0021	(2, 3)	1001	(3, 2)	1111	(5, 4)
0022	(1, 4)	1002	(3, 3)	1112	(5, 5)
0023	(0, 5)	1003	(2, 4)	1113	(2, 6)
0100	(2, 1)	1010	(4, 2)	1120	(4, 4)
0101	(2, 2)	1011	(4, 3)	1121	(4, 5)
0102	(2, 3)	1012	(4, 4)	1122	(3, 6)
0103	(1, 4)	1013	(2, 5)	1123	(0, 7)

The following matrix now gives the distribution of the numbers type-wise:

u/w	0	1	2	3	4	5	6	7	Row-sums
0	1	1	1	1	1	1	1	1	8
1	0	1	1	1	2	1	1	0	7
2	0	1	2	2	1	3	1	0	10
3	0	1	2	3	2	1	1	0	10
4	0	0	2	3	4	1	0	0	10
5	0	0	0	1	1	1	0	0	3
Column-sums:	1	4	8	11	11	8	4	1	48

The row-sums are of particular importance in the divisor problem mentioned in the introduction.

Given an arbitrary prime p and a sequence $D_k = (d_1, d_2, \ldots, d_k)$ of positive integers, it is known that the number of right divisors in the Hermite normal form, of the diagonal matrix

$$\text{diag}\,(p^{d_1},\ p^{d_2}, \ldots, p^{d_k})$$

is given by a polynomial in p such as

$$Q(p) = \sum_{u=0}^{h} c_u p^u;$$

where h depends on D_k. We find that the degree h of the polynomial is precisely the same as the highest degree achieved by the numbers representable and represented in the multi-scalar system generated by D_k, and the c_u's are nothing but the u-row sums in the distribution matrix.

For example, the polynomial in the present case is:

$$Q(p) = 8 + 7p + 10p^2 + 10p^3 + 10p^4 + 3p^5,$$

and it is the number of the said right divisors of

$$\text{diag}\,(p, p, p^2, p^3).$$

5. Number of Numbers Which have a Given Weight

Theorem 1 — For any given sequence D_k, the number of numbers which have a given weight w, is the coefficient of x^w in the expansion of the product:

$$\prod_{i=1}^{k} (1 + x + x^2 + \ldots + x^{d_i}).$$

Proof : The number of numbers with the given weight w is the same as the number of solutions of the equation

$$a_1 + a_2 + \dots + a_k = w \qquad \dots(5.1)$$

with $0 \leqslant a_i \leqslant d_i, \quad i \leqslant k$.

This is evidently the coefficient of the wth power of x in the product given in the statement of the theorem.

In our example in the preceding section, the product is

$$(1 + x)(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3).$$

On expansion it turns out to be

$$= 1 + 4x + 8x^2 + 11x^3 + 11x^4 + 8x^5 + 4x^6 + x^7.$$

The coefficients are the column sums in our presentation of the distribution matrix.

Since the polynomials involved in the product are reciprocal polynomials, the coefficients in the expanded product are naturally the same from the beginning and the end.

This fact follows also from the 1–1 correspondence between the numbers of weight w and their complements which are all of weight $S - w$, where

$$S = d_1 + d_2 + \dots + d_k.$$

(This result provides a good check on the distribution matrix.)

6. Extension of the Distribution Matrix

Assume that the distribution matrix for a certain sequence $(d_1, d_2, \dots, d_k)$ is already available. Then the problem is to find how the distribution matrix for the sequence

$$(d_1, d_2, \dots, d_k, d_{k+1}); \text{ with } d_{k+1} \geqslant d_k;$$

could be constructed.

If some number

$$a_1 a_2 \dots a_k$$

is of the type (u, w), our problem is really to find the type of the number:

$$a_1 a_2 \dots a_k a_{k+1} \qquad \dots(6.1)$$

for each a_{k+1} from 0 to d_{k+1}.

Evidently, the degree of the number in (6.1) is

$$u + \text{the degree of } a_{k+1}$$

$$= u + W(b_{k+1})(a_1 a_2 \dots a_k); \qquad \dots(6.2)$$

and its weight is

$$w + a_{k+1}. \qquad \dots(6.3)$$

We have thus to determine the type of each of the

$$(d_1 + 1)(d_2 + 1) \dots (d_{k+1} + 1)$$

numbers anew.

On the other hand if the extension is made on the left with the addition of a $d_0 (\leqslant d_1)$ to the sequence $(d_1, d_2, \dots, d_k)$; our task is very much simpler.

The weight of the number

$$a_0 a_1 a_2 \dots a_k \qquad \dots(6.4)$$

is $w + a_0$, and its degree is

$$u + W(a_0)(b_1 b_2 \dots b_k). \qquad \dots(6.5)$$

For $a_0 = 0$, there is no change in the type of (6.4), and the values of a_0 we need consider are only those for which $1 \leqslant a_0 \leqslant d_0$.

We leave it to the reader to first obtain the distribution matrix for the sequence $d_1 = 2$, $d_2 = 3$ and extend it by adding $d_3 = 5$ to the sequence on the right and then obtain the distribution matrix for the sequence $d_2 = 3$, $d_3 = 5$ and extend it by adding $d_1 = 2$ on the left of the sequence. The final result in either case will be:

u/w	0	1	2	3	4	5	6	7	8	9	10	
0	1	1	1	1	1	1	1	1	1	1	1	11
1	0	1	1	1	1	1	2	2	1	1	0	11
2	0	1	2	2	2	2	1	1	2	1	0	14
3	0	0	1	2	2	2	2	2	1	0	0	12
4	0	0	1	2	3	3	2	1	1	0	0	13
5	0	0	0	1	2	3	3	2	0	0	0	11
	1	3	6	9	11	12	11	9	6	3	1	72

7. The Case $k = 2$

Let the sequence of d's consist of only two d's d_1 and d_2, with $1 \leqslant d_1 \leqslant d_2$. Then we prove

Theorem 2 — For $0 \leqslant u \leqslant d_1$,

$$N(u, w) = 1 \text{ if } u \leqslant w \leqslant d_1 + d_2 - u;$$
$$= 0 \text{ otherwise.}$$

Proof : Two cases arise:

(i) If $a_1 a_2$ is a number of the type (u, w) and $a_1 \leqslant b_2$, then we have

$$a_1 = u \text{ and consequently } a_2 = w - u.$$

Since a_2 is necessarily nonnegative, we must have

$$w \geqslant u. \qquad \ldots(7.1)$$

Also $a_1 \leqslant d_1$, therefore u must be $\leqslant d_1$. ...(7.2)

On the other hand,

(ii) If $b_2 < a_1$, then we have

$$u = b_2 = d_2 - a_2.$$

This means that

$$a_2 = d_2 - u \text{ and consequently } a_1 = w - a_2 = u + w - d_2.$$

Since $0 \leqslant a_1 \leqslant d_1$, we must have

$$u + w \leqslant d_1 + d_2. \qquad \ldots(7.3)$$

Whenever the conditions in (7.1), (7.2), (7.3) are satisfied there is just one number of the type (u, w) otherwise there is none and the theorem is proved.

As a direct consequence of our theorem, the entries in the row for any given u for which $0 \leqslant u \leqslant d_1$ in the distribution matrix will be

u zero's followed by $(d_1 + d_2 + 1 - 2u)$ ones followed by u zero's.

The row sums will therefore be given by the arithmetical series:

$$d_1 + d_2 + 1,\ d_1 + d_2 - 1,\ \ldots,\ d_2 - d_1 + 1.$$

Example — For $d_1 = 4$, $d_2 = 7$, the distribution matrix is:

u/w	0	1	2	3	4	5	6	7	8	9	10	11	
0	1	1	1	1	1	1	1	1	1	1	1	1	12
1	0	1	1	1	1	1	1	1	1	1	1	0	10
2	0	0	1	1	1	1	1	1	1	1	0	0	8
3	0	0	0	1	1	1	1	1	1	0	0	0	6
4	0	0	0	0	1	1	1	1	0	0	0	0	4
	1	2	3	4	5	5	5	5	4	3	2	1	40

8. The Case $d_1 = d_2 = \ldots = d_k = m$

Of special importance is the case where each d in the sequence is equal to m, $m \geqslant 1$, $k \geqslant 2$.

Let $M(u, w)$ denote in this case the number of numbers of degree u and weight w. Then we prove

Theorem 3 — $M(u, w) = M(u, km - w)$.

(This means that the entries in any row of the distribution matrix are equal from the beginning and the end).

PROOF : Let

$$a_1a_2 \ldots a_k$$

be any number of the type (u, w). Then we show that the number

$$b_kb_{k-1} \ldots b_2b_1 \quad \ldots(8.1)$$

with $b_j = m - a_j$, $j \leqslant k$; is of degree u and weight $km - w$.

The second statement needs no proof. The first statement is proved by noting that

$$u = \sum_{j=2}^{k} W(b_j)\,(a_1a_2 \ldots a_{j-1}) \quad \ldots(8.2)$$

$$= \sum_{j=2}^{k} \sum_{t=1}^{j-1} \min\,(b_j, a_t);$$

while the degree of the number in (8.1)

$$= \sum_{t=1}^{k-1} W(a_t)\,(b_{t+1}\, b_{t+2} \ldots b_k)$$

$$= \sum_{t=1}^{k-1} \sum_{j=t+1}^{k} \min\,(a_t, b_j).$$

This is the same as

$$\sum_{j=2}^{k} \sum_{t=1}^{j-1} \min\,(b_j, a_t).$$

Hence the theorem follows readily.

9. INTERCHANGE OF CONSECUTIVE ELEMENTS

Consider the two numbers

$$a_1a_2 \ldots a_{k-2}a_{k-1}a_k \quad \ldots(9.1)$$

and

$$a_1a_2 \ldots a_{k-2}a_ka_{k-1} \quad \ldots(9.2)$$

which differ only in having the digits a_k and a_{k-1} interchanged. Assume that

$$m \geqslant \max\,(a_1, a_2, \ldots, a_k) \text{ and } a_k > a_{k-1}.$$

Then we prove

Theorem 4 — The degree of the number in (9.2) — the degree of the number in (9.1) $= a_k - a_{k-1}$.

PROOF : The degree of a_k in (9.1) $= W(b_k)\ (a_1a_2 \ldots a_{k-1})$;

The degree of a_{k-1} in (9.1) $= W(b_{k-1})\ (a_1a_2 \ldots a_{k-2})$.

On the other hand, the degree of a_k in (9.2) $= W(b_k)\ (a_1a_2 \ldots a_{k-2})$; and the degree of a_{k-1} in (9.2) $= W(b_{k-1})\ (a_1a_2 \ldots a_{k-2}a_k)$. Since the degree of any other element in the two numbers is the same for both, we have the degree of the second number — the degree of the first

$$= W(b_{k-1})\ (a_k) - W(b_k)\ (a_{k-1})$$

$$= \min\ (b_{k-1}, a_k) - \min\ (b_k, a_{k-1}). \qquad \ldots(9.3)$$

Since $a_k - a_{k-1} = b_{k-1} - b_k$, it follows that

$a_k < b_{k-1}$ if and only if $a_{k-1} < b_k$;

and $a_k > b_{k-1}$ if and only if $a_{k-1} > b_k$.

Hence, the difference in (9.3) $= a_k - a_{k-1} = b_{k-1} - b_k$.

The argument applies if any two consecutive elements in a number are interchanged. Note that if two numbers differ only in having two consecutive elements interchanged, then the greater of the two has the greater degree and the difference in the degrees is the same as the difference between the digits interchanged.

As an application, with $m \geqslant 5$, let us consider the set of numbers obtained by permuting the digits of the number 02042. Assume that the degree of the smallest number 00224 of the set is u and let us find the degrees of the other members of the set.

The set has 30 members in all. These are

(1)	00224	(2)	00242	(3)	00422	(4)	02024	(5)	02042
(6)	02204	(7)	02240	(8)	02402	(9)	02420	(10)	04022
(11)	04202	(12)	04220	(13)	20024	(14)	20042	(15)	20204
(16)	20240	(17)	20402	(18)	20420	(19)	22004	(20)	22040
(21)	22400	(22)	24002	(23)	24020	(24)	24200	(25)	40022
(26)	40202	(27)	40220	(28)	42002	(29)	42020	(30)	42200

In the following scheme, these numbers are so arranged that two neighbours on the same route differ only in having two consecutive digits interchanged and the theorem of this section is readily applicable.

Route 1 : From (1) through (2), (3), (10), (25), (26), (28), (29) to (30);

Route 2 : From (1) through (4), (13), (14), (17), (22), (23) to (24);

Route 3 : From (4) through (6), (15), (19), (20) to (21);

Route (a) : From (2) through (5), (8) to (9);

Route (b) : From (6) to (7);

Route (c) : From (10) via (11) to (12);

Route (d) : From (15) to (16);

Route (e) : From (17) to (18);

Route (f) : From (26) to (27).

The following table now gives the degrees of the numbers listed:

(1)	00224	u
(2)	00242	$u + 2$
(3)	00422	$u + 4$
(10)	04022	$u + 8$
(25)	40022	$u + 12$
(26)	40202	$u + 14$
(28)	42002	$u + 16$
(29)	42020	$u + 18$
(30)	42200	$u + 20$

(1)	00224	u
(4)	02024	$u + 2$
(13)	20024	$u + 4$
(14)	20042	$u + 6$
(17)	20402	$u + 10$
(22)	24002	$u + 14$
(23)	24020	$u + 16$
(24)	24200	$u + 18$

(4)	02024	$u + 2$
(6)	02204	$u + 4$
(15)	20204	$u + 6$
(19)	22004	$u + 8$
(20)	22040	$u + 12$
(21)	22400	$u + 16$

(2)	00242	$u + 2$
(5)	02042	$u + 4$
(8)	02402	$u + 8$
(9)	02420	$u + 10$

(6)	02204	$u + 4$
(7)	02240	$u + 8$

(10)	04022	$u + 8$
(11)	04202	$u + 10$
(12)	04220	$u + 12$

(15)	20204	$u + 6$
(16)	20240	$u + 10$

(17)	20402	$u + 10$
(18)	20420	$u + 12$

(26)	40202	$u + 14$
(27)	40220	$u + 16$

For a suitable m, let S denote the set of numbers formed by permuting the elements of the number

$$a_1 a_2 \dots a_k.$$

Let u denote the degree of the smallest member of S and U that of the largest. Then we prove

Theorem 5 — degree of $a_1 a_2 \dots a_k$ + degree of $a_k a_{k-1} \dots a_1 = u + U$.

PROOF : If we interchange any two consecutive elements say a_j and a_{j-1} in each of the two numbers, the degree of one of the numbers gains as much as that of the other loses. Moreover, the interchanges that will transform one of the numbers into the greatest member of the set will automatically transform the other into the smallest member of the set. Hence the theorem. This means that if

$$\text{the degree of } a_1 a_2 \dots a_k = u + t, \quad 0 \leqslant t \leqslant U - u;$$

then the degree of $a_k a_{k-1} \dots a_1 = U - t$.

This implies, if $T(r)$ denotes the number of members of S which have the degree r, then

$$T(u + t) = T(U - t). \qquad \dots(9.4)$$

10. Some More Applications of Theorem 4

(i) Given m and k, one can ask what is the highest degree that the numbers of a given weight w can attain?

Take for example $m = 5$, $k = 7$ and $w = 14$, then the highest degree is attained by the number (5540000).

The degree attained is 58.

(ii) Similarly one can ask among the system of numbers generated by a given m for a given k, what is the highest degree attained by any number?

It is easy to see that the highest degree is attained by the number

$$a_1 a_2 \dots a_k$$

where

$$a_1 = a_2 = \dots = a_t = m, \text{ with } t = [(k + 1)/2];$$

$$a_{t+1} = a_{t+2} = \dots = a_k = 0.$$

For k odd, $t = [k/2]$ gives another number with the highest degree. The highest degree attained is given by

$$m \cdot [k^2/4]. \qquad \dots(10.1)$$

(Here the square bracket denotes the greatest integer function.)

11. The Case $m = 2$: Some Basic Relations

Let

$$a_1 a_2 \dots a_k \qquad \dots(11.1)$$

be a number of the type (u, w). Among the digits $a_1, a_2, \dots, a_k$ of the number suppose exactly v are different from zero, i.e. they are either ones or twos.

Let us indicate this fact by saying that (11.1) is a number of the type (u, v, w).

If we now wish to extend our system from k digited numbers to $(k + 1)$ digited numbers by introducing $d_{k+1} = 2$, then we need consider only how the type of (11.1) is affected if we insert one of the digits 0, 1 or 2 on the right of (11.1).

It will be readily seen that insertion of zero changes the type of (11.1) from (u, v, w) to $(u + w, v, w)$. The insertion of one gives a number of the type $(u + v, v + 1, w + 1)$, while insertion of two produces a number of the type $(u, v + 1, w + 2)$.

Given the distribution matrix for some k, in the distribution matrix for $(k + 1)$, the numbers which have a zero on the right are taken care of if we push the column headed w in the matrix for k, as many as w places down. Similarly, the numbers which have 2 on the right are accounted for, if we move the entries in the matrix for k, two places to the right. The main problem we have, therefore, is to find how the numbers which have 1 on the right can be taken care of. The distribution matrix for such numbers will be designated an auxiliary matrix for $(k + 1)$. To clarify the whole situation, let us take $k = 4$ and calculate its column for say $w = 4$.

The only relevant partitions of w that is 4 into atmost k that is 4 parts are 2 2 0 0; 2 1 1 0, and 1 1 1 1. The permutations of these numbers provide all the numbers of weight 4. We list these noting the degree of each against it. Note that $v = 2$ for the numbers arising from the partition 2 2 0 0; $v = 3$ for the numbers arising from the partition 2 1 1 0; and $v = 4$ for those arising from the partition 1 1 1 1.

$v = 2$		$v = 3$		$v = 4$	
0022	0	0112	1	1111	6
0202	2	0121	2		
0220	4	0211	3		
2002	4	1012	2		
2020	6	1021	3		
2200	8	1102	3		
		1120	5		
		1201	5		
		1210	6		
		2011	5		
		2101	6		
		2110	7		

The distribution matrix for these numbers is:

u/v	2	3	4	
0	1	0	0	1
1	0	1	0	1
2	1	2	0	3
3	0	3	0	3
4	2	0	0	2
5	0	3	0	3
6	1	2	1	4
7	0	1	0	1
8	1	0	0	1
	6	12	1	19

...(11.2)

The row sums in the above matrix provide the entries in the matrix for $k = 4$, for the column $w = 4$.

For the auxiliary matrix for $k+1=5$, we add the digit 1 to the right of each of the numbers listed earlier.

Recall that by this insertion of 1, the degree of the number concerned increases by the corresponding value of v; v increases by 1 and becomes $(v+1)$ and so does w change to $(w+1)$.

To obtain the entries in the auxiliary matrix for $k=5$, $w=5$, we push the columns of (11.2) v places down getting the matrix:

u/v	3	4	5	
0				
1				
2	1			1
3	0	0		0
4	1	1	0	2
5	0	2	0	2
6	2	3	0	5
7	0	0	0	0
8	1	3	0	4
9	0	2	0	2
10	1	1	1	3
	6	12	1	19

The row sums provide the entries for the auxiliary matrix. While the entries in the columns for different v's satisfy the symmetry relation of Theorem 5, the row sums do not do so. Entries for the other columns of the auxiliary matrix can be computed in the same manner. The auxiliary matrix for $k=5$ will then be found to be given by

u/w	0	1	2	3	4	5	6	7	8	9	10	
0		1										1
1			1	1								2
2			1	0	1	1						3
3			1	2	2	0	1	1				7
4			1	1	1	2	2	0	1	1		9
5				3	2	2	1	2	1			11
6				1	3	5	2	1	1			13
7				2	2	0	3	3	1			11
8					3	4	2	1				10
9					2	2	3	2				9
10						3	2					5
		1	4	10	16	19	16	10	4	1		81

Before proceeding any further, we must prove a property of the auxiliary matrix. This is stated in

Theorem 6 — Let $A(u, w)$ denote the entry in the column headed w and against the degree u, in the auxiliary matrix for any k.

Then $A(u, w) = A(u + k - w, 2k - w)$.

PROOF : For any k, let

$$a_1 a_2 \ldots a_{k-1}\, a_k \text{ with } a_k = 1$$

be any number of the type (u, v, w).
Then the number

$$a_1 a_2 \ldots a_{k-1} \qquad \ldots(11.3)$$

must be of the type $(u - v + 1, v - 1, w - 1)$.
Among the a's in (11.3), therefore, there must be

$(w - v)$ two's, $(v - 1) - (w - v) = 2v - w - 1$ one's, and $(k - v)$ zero's.

This implies that the number

$$b_{k-1} b_{k-2} \ldots b_1 b_k \qquad \ldots(11.4)$$

where $b_k = 2 - a_k = 1$, has exactly $(k - v)$ 2's, $(2v - w)$ 1's, and $(w - v)$ 0's among its elements.

Since the numbers

$$a_1 a_2 \ldots a_{k-1} \text{ and } b_{k-1} b_{k-2} \ldots b_1$$

are of the same degree as already proved, the latter number must be of the type $(u - v + 1, k + v - w - 1, 2k - w - 1)$.

This implies that the number in (11.4) is of the type

$$(u + k - w, k + v - w, 2k - w).$$

Thus corresponding to every number of the type (u, v, w) ending in 1, there is just one number of the type $(u + k - w, k + v - w, 2k - w)$ which ends in 1. This proves the theorem that

$$A(u, w) = A(u + k - w, 2k - w).$$

This means that the entries in the auxiliary matrix for any k, in the columns headed w and $(2k - w)$ are the same though they start at different levels. See for example the auxiliary matrix for $k = 5$ given earlier in this section.

12. THE DISTRIBUTION MATRIX FOR $(k + 1)$, $m = 2$

The procedure having already been discussed in the previous section, we now show how we can obtain the distribution matrix for $k = 5$ from that for $k = 4$. The distribution matrix for $k = 4$ is:

u/w	0	1	2	3	4	5	6	7	8	
0	1	1	1	1	1	1	1	1	1	9
1	0	1	1	2	1	2	1	1	0	9
2	0	1	2	1	3	1	2	1	0	11
3	0	1	2	3	3	3	2	1	0	15
4	0	0	2	3	2	3	2	0	0	12
5	0	0	1	2	3	2	1	0	0	9
6	0	0	1	3	4	3	1	0	0	12
7	0	0	0	1	1	1	0	0	0	3
8	0	0	0	0	1	0	0	0	0	1

Pushing down each column of the matrix for $k = 4$, w steps, we obtain the matrix:

u/w	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	0	1	1						
3	0	1	1	1					
4	0	1	2	2	1				
5	0	0	2	1	1	1			
6	0	0	2	3	3	2	1		
7	0	0	1	3	3	1	1	1	
8	0	0	1	2	2	3	2	1	1
9		0	0	3	3	3	2	1	0
10			0	1	4	2	2	1	0
11				0	1	3	1	0	0
12					1	1	1	0	0

Next shifting each row of the matrix for $k = 4$, two places to the right, we get the matrix:

u/w	0	1	2	3	4	5	6	7	8	9	10
0			1	1	1	1	1	1	1	1	1
1			0	1	1	2	1	2	1	1	0
2			0	1	2	1	3	1	2	1	0
3			0	1	2	3	3	3	2	1	0
4			0	0	2	3	2	3	2	0	0
5			0	0	1	2	3	2	1	0	0
6			0	0	1	3	4	3	1	0	0
7			0	0	0	1	1	1	0	0	0
8			0	0	0	0	1	0	0	0	0

And the auxiliary matrix for $k = 5$ has already been recorded in the preceding section. Adding the three matrices in the usual manner, we get the matrix:

u/w	0	1	2	3	4	5	6	7	8	9	10	
0	1	1	1	1	1	1	1	1	1	1	1	11
1	0	1	1	2	1	2	1	2	1	1	0	12
2	0	1	2	1	3	2	3	1	2	1	0	16
3	0	1	2	4	4	3	4	4	2	1	0	25
4	0	1	3	3	4	5	4	3	3	1	0	27
5	0	0	2	4	4	5	4	4	2	0	0	25
6	0	0	2	4	7	10	7	4	2	0	0	36
7	0	0	1	5	5	2	5	5	1	0	0	24
8	0	0	1	2	5	7	5	2	1	0	0	23
9	0	0	0	3	5	5	5	3	0	0	0	21
10	0	0	0	1	4	5	4	1	0	0	0	15
11	0	0	0	0	1	3	1	0	0	0	0	5
12	0	0	0	0	1	1	1	0	0	0	0	3
	1	5	15	30	45	51	45	30	15	5	1	243

This is the distribution matrix for $k = 5$.

It may be pointed out here that while the other columns get checked automatically, it is the central column alone that would need checking by the method described in section 11.

The distribution matrices for $k = 6$ and 7 and some tables are given below.

Distribution Matrices

$m = 2, \quad k = 6$

u/w	0	1	2	3	4	5	6	7	8	9	10	11	12	
0	1	1	1	1	1	1	1	1	1	1	1	1	1	13
1		1	1	2	1	2	1	2	1	2	1	1		15
2		1	2	1	3	2	3	2	3	1	2	1		21
3		1	2	4	4	4	5	4	4	4	2	1		35
4		1	3	4	5	5	6	5	5	4	3	1		42
5		1	3	4	6	7	6	7	6	4	3	1		48
6			3	6	8	12	10	12	8	6	3			68
7			2	6	8	9	10	9	8	6	2			60
8			2	5	9	11	12	11	9	5	2			66
9			1	6	9	12	18	12	9	6	1			74
10			1	5	10	13	14	13	10	5	1			72
11				2	8	13	8	13	8	2				54
12				3	7	8	14	8	7	3				50
13				1	5	10	10	10	5	1				42
14					4	8	9	8	4					33
15					1	5	8	5	1					20
16					1	3	4	3	1					12
17						1	1	1						3
18							1							1
	1	6	21	50	90	126	141	126	90	50	21	6	1	729

$m = 2,\ k = 7$

u/w	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	15
1		1	1	2	1	2	1	2	1	2	1	2	1	1		18
2		1	2	1	3	2	3	2	3	2	3	1	2	1		26
3		1	2	4	4	4	5	5	5	4	4	4	2	1		45
4		1	3	4	5	6	7	5	7	6	5	4	3	1		57
5		1	3	5	7	7	8	9	8	7	7	5	3	1		71
6		1	4	6	10	14	12	14	12	14	10	6	4	1		108
7			3	8	9	11	13	17	13	11	9	8	3			105
8			3	6	12	17	17	17	17	17	12	6	3			127
9			2	9	13	16	24	23	24	16	13	9	2			151
10			2	7	14	23	27	23	27	23	14	7	2			169
11			1	7	14	19	23	30	23	19	14	7	1			158
12			1	6	15	23	28	23	28	23	15	6	1			169
13				5	12	19	27	42	27	19	12	5				168
14				2	13	23	29	27	29	23	13	2				161
15				3	10	18	29	32	29	18	10	3				152
16				1	7	20	23	21	23	20	7	1				123
17					5	11	18	27	18	11	5					95
18					4	13	21	21	21	13	4					97
19					1	8	15	17	15	8	1					65
20					1	5	12	14	12	5	1					50
21						3	8	12	8	3						34
22						1	4	5	4	1						15
23							1	3	1							5
24							1	1	1							3
	1	7	28	77	161	266	357	393	357	266	161	77	28	7	1	2187

TABLES

$m = 2,\ k \leqslant 6$

$v = 0,\ w = 0$	u
0	0

$v = 1,\ w = 1$	u
1	0
10	1
100	2
1000	3
10000	4
100000	5

$v = 1,\ w = 2$	u
2	0
20	2
200	4
2000	6
20000	8
200000	10

$v = 2,\ w = 2$	u
11	1
101	2
110	3
1001	3
1010	4
1100	5
10001	4
10010	5
10100	6
11000	7
100001	5
100010	6
100100	7
101000	8
110000	9

$v = 2,\ w = 3$	u
12	0
21	1
102	1
120	3
201	3
210	4
1002	2
1020	4
1200	6
2001	5
2010	6
2100	7
10002	3
10020	5
10200	7

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12000	9
20001	7
20010	8
20100	9
21000	10
100002	4
100020	6
100200	8
102000	10
120000	12
200001	9
200010	10
200100	11
201000	12
210000	13

$v = 3, w = 3$

111	3
1011	4
1101	5
1110	6
10011	5
10101	6
10110	7
11001	7
11010	8
11100	9
100011	6
100101	7
100110	8
101001	8
101010	9
101100	10
110001	9
110010	10
110100	11
111000	12

$v = 2, w = 4$

22	0
202	2
220	4
2002	4
2020	6
2200	8
20002	6
20020	8
20200	10
22000	12
200002	8
200020	10
200200	12
202000	14
220000	16

$v = 3, w = 4$

112	1
121	2
211	3
1012	2
1021	3
1102	3
1120	5
1201	5
1210	6
2011	5
2101	6
2110	7
10012	3
10021	4
10102	4
10120	6
10201	6
10210	7
11002	5
11020	7
11200	9
12001	8
12010	9
12100	10
20011	7
20101	8
20110	9
21001	9
21010	10
21100	11
100012	4
100021	5
100102	5
100120	7
100201	7
100210	8
101002	6
101020	8
101200	10
102001	9
102010	10
102100	11
110002	7
110020	9
110200	11
112000	13
120001	11
120010	12
120100	13
121000	14
200011	9
200101	10
200110	11
201001	11
201010	12
201100	13
210001	12
210010	13
210100	14
211000	15

$v = 4, w = 4$

1111	6
10101	7
11011	8
11101	9
11110	10
100111	8
101011	9
101101	10
101110	11
110011	10
110101	11
110110	12
111001	12
111010	13
111100	14

$v = 3, w = 5$

122	0
212	1
221	2
1022	1
1202	3
1220	5
2012	3
2021	4
2102	4
2120	6
2201	6
2210	7
10022	2
10202	4
10220	6
12002	6
12020	8
12200	10
20012	5
20021	6
20102	6
20120	8
20201	8
20210	9
21002	7
21020	9
21200	11
22001	10
22010	11
22100	12
100022	3
100202	5
100220	7
102002	7
102020	9
102200	11
120002	9
120020	11
120200	13
122000	15
200012	7
200021	8
200102	8
200120	10
200201	10
200210	11
201002	9
201020	11
201200	13
202001	12
202010	13
202100	14
210002	10
210020	12
210200	14
212000	16
220001	14
220010	15

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220100	16
221000	17

$v = 4, w = 5$

1112	3
1121	4
1211	5
2111	6
10112	4
10121	5
10211	6
11012	5
11021	6
11102	6
11120	8
11201	8
11210	9
12011	8
12101	9
12110	10
20111	8
21011	9
21101	10
21110	11
100112	5
100121	6
100211	7
101012	6
101021	7
101102	7
101120	9
101201	9
101210	10
102011	9
102101	10
102110	11
110012	7
110021	8
110102	8
110120	10
110201	10
110210	11
111002	9
111020	11
111200	13
112001	12
112010	13
112100	14
120011	11
120101	12
120110	13
121001	13
121010	14
121100	15
200111	10
201011	11
201101	12
201110	13
210011	12
210101	13
210110	14
211001	14
211010	15
211100	16

$v = 5, w = 5$

11111	10
101111	11
110111	12
111011	13
111101	14
111110	15

$v = 3, w = 6$

222	0
2022	2
2202	4
2220	6
20022	4
20202	6
20220	8
22002	8
22020	10
22200	12
200022	6
200202	8
200220	10
202002	10
202020	12
202200	14
220002	12
220020	14
220200	16
222000	18

$v = 4, w = 6$

1122	1
1212	2
1221	3
2112	3
2121	4
2211	5
10122	2
10212	3
10221	4
11022	3
11202	5
11220	7
12012	5
12021	6
12102	6
12120	8
12201	8
12210	9
20112	5
20121	6
20211	7
21012	6
21021	7
21102	7
21120	9
21201	9
21210	10
22011	9
22101	10
22110	11
100122	3
100212	4
100221	5
101022	4
101202	6
101220	8
102012	6
102021	7
102102	7
102120	9
102201	9
102210	10
110022	5
110202	7
110220	9
112002	9
112020	11
112200	13
120012	8
120021	9
120102	9
120120	11
120201	11
120210	12
121002	10
121020	12
121200	14
122001	13
122010	14
122100	15
200112	7
200121	8
200211	9
201012	8
201021	9
201102	9
201120	11
201201	11
201210	12
202011	11
202101	12
202110	13
210012	9
210021	10
210102	10
210120	12
210201	12
210210	13
211002	11
211020	13
211200	15
212001	14
212010	15
212100	16
220011	13
220101	14
220110	15
221001	15
221010	16
221100	17

$v = 5, w = 6$

11112	6
11121	7
11211	8

(Continued next page)

12111	9	111102	10	121110	15
21111	10	111120	12	201111	12
101112	7	111201	12	210111	13
101121	8	111210	13	211011	14
101211	9	112011	12	211101	15
102111	10	112101	13	211110	16
110112	8	112110	14		
110121	9	120111	12	$v = 6, w = 6$	
110211	10	121011	13		
111012	9	121101	14	111111	15
111021	10				

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