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ON A CLASS OF LINEAR POSITIVE OPERATORS

B. KUNWAR

Department of Mathematics, Indian Institute of Technology, Kanpur 208016

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A new class of linear positive operators has been introduced, which contains a number of well known linear positive operators, for examples the Gamma operators of Müller, the modified Post-Widder and Post-Widder operators, etc., as special cases. Some of the basic approximation properties have been studied for this class.

1. INTRODUCTION

During the past three decades a number of classes and sequences of linear positive operators both summation type and those defined by certain integrals, have been introduced and studied by various authors. Some of the well known operators of the latter type are the Gamma operators of Müller (1967), Post-Widder operators (Widder 1946), modified Post-Widder operators (May 1976), Gauss-Weirstrass integrals (Rathore 1973), convolution operators (Shapiro 1969) and the operators studied in Baskakov (1968), De Vore (1970), Leviatan (1972), Micchelli (1969) and Sikkema and Rathore (1976).

In the present paper we introduce a new class of linear positive operators which contains a number of well known linear operators as special cases. Moreover, we study some of the basic approximation properties of this class.

Throughout the paper $\mathbf{R}^+$ denotes the interval $(0, \infty)$, $\langle a, b \rangle$, the open interval containing $[a, b] \subset \mathbf{R}^+$, $\chi_{\delta,x}(\chi^c_{\delta,x})$ the characteristic function of the interval $(x - \delta, x + \delta)$ $(\mathbf{R}^+ \setminus (x - \delta, x + \delta))$. The spaces $M(\mathbf{R}^+)$, $M_b(\mathbf{R}^+)$, Loc $(\mathbf{R}^+)$, $L^1(\mathbf{R}^+)$ respectively denote the sets of complex-valued measurable, bounded and measurable, locally integrable and Lebesgue-integrable functions on $\mathbf{R}^+$.

2. THE OPERATORS

In this section we first introduce the operators T_λ and give more elementary properties of the same.

Definition — Let $G \in M(\mathbf{R}^+)$ be a non-negative function satisfying:

(i) $G(u)$ is continuous at $u = 1$,

(ii) for each $\delta > 0$, $\| \chi^c_{\delta,1} G \|_\infty < G(1)$, and

(iii) there exist $\theta_1, \theta_2 > 0$ such that

$$(u^{-\theta_1} + u^{\theta_2})\, G(u) \in M_b(\mathbf{R}^+).$$

Such a function G is called an 'admissible' kernel function. The set of kernel functions will be denoted by $T(\mathbf{R}^+)$.

Definition — Let $G \in T(\mathbf{R}^+)$ and α be a real number. Then for $\lambda, x \in \mathbf{R}^+$ and $f \in M(\mathbf{R}^+)$ we define

$$T_\lambda(f; x) = T_\lambda^{G,\alpha}(f; x) = \frac{x^{\alpha-1}}{a(\lambda)} \int_0^\infty u^{-\alpha} f(u)\, G^\lambda(xu^{-1})\, du \qquad \ldots(2.1)$$

where

$$a(\lambda) = \int_0^\infty u^{\alpha-2} G^\lambda(u)\, du \qquad \ldots(2.2)$$

whenever the above integrals exist.

In the sequel we shall assume G and α to be arbitrary, but fixed. Thus, the superscripts G and α will be dropped and the operators $T_\lambda^{G,\alpha}$ will be written simply as T_λ.

Definition — Let $\Omega(> 1)$ be a continuous function defined on $\mathbf{R}^+$. We call Ω a bounding function for G, if for each compact $K \subset \mathbf{R}^+$ there exist positive numbers λ_K and M_K such that

$$T_{\lambda_K}(\Omega : x) < M_K,\ x \in K. \qquad \ldots(2.3)$$

It is clear, that if $G \in T(\mathbf{R}^+)$ then $\Omega(u) = u^{-q} + u^p$ is a bounding function for G, where $p, q > 0$ are arbitrary.

The notion of a bounding function (Rathore 1974) enables us to obtain results in a uniform setup, which at the same time, are applicable for a general $G \in T(\mathbf{R}^+)$. Indeed, here we are concerned with results which are plausible for functions having as high an order of unboundedness (when $u \to 0$ or ∞) as possible.

For a bounding function Ω, we define $D_\Omega = \{f \in \text{Loc}\,(\mathbf{R}^+)$ such that

$$\overline{\lim_{u\to 0}}\, f(u)/\Omega(u) \text{ and } \overline{\lim_{u\to\infty}}\, f(u)/\Omega(u) \text{ exist}\}.$$

Several well known operators are particular cases of the operators T_λ:

(i) Taking $G(u) = ue^{-u}$, $\alpha = 2$ and putting $\lambda = n$, the operator T_λ reduces to the Gamma operator G_n of Müller (1967) defined by,

$$G_n(f; x) = \frac{x^{n+1}}{n\,!} \int_0^\infty u^n e^{-ux} f\left(\frac{n}{u}\right) du. \qquad \text{...(2.4)}$$

It is easy to see that $G(u) \in T(\mathbf{R}^+)$. Moreover, $\Omega(u) = u^b + e^{a/u}$ is a bounding function for G, for all $a, b > 0$.

(ii) With $G(u) = u^{-1}, e^{-u^{-1}}$ $\lambda = n, \alpha = 1$, the operator T_λ becomes the Post-Widder operator S_n^1 (May 1976), defined by,

$$S_n^1(f; t) = \frac{1}{(n-1)!} \left(\frac{n}{t}\right)^n \int_0^\infty e^{-nu/t} u^{n-1} f(u)\, du. \qquad \text{...(2.5)}$$

In this case also, $G(u) \in T(\mathbf{R}^+)$ and moreover $\Omega(u) = u^{-b} + e^{au}$, $(a, b > 0)$ is a bounding function for G.

(iii) With $\alpha = 0$, $\lambda = k$ and $G(u)$ the same as in (ii), the operator T_λ gives rise to the operators $L_{k,t}$ (Widder 1946) defined by,

$$L_{k,t}[f; x] = \frac{1}{k} \left(\frac{k}{t}\right)^{k+1} \int_0^\infty e^{-ku/t} u^k f(u)\, du. \qquad \text{...(2.6)}$$

In each of the above three cases, we further notice that $G''(1) \neq 0$.

(iv) The more general operator L_λ (Kunwar 1979) defined by

$$L_\lambda(f; x) = \frac{p\lambda^{\lambda+(\alpha-1)/p}}{\Gamma(\lambda + (\alpha - 1/p))} x^{\lambda p+\alpha-1} \int_0^\infty u^{-\lambda p-\alpha} e^{-\lambda(x/u)^p} f(u)\, du \qquad \text{...(2.7)}$$

$(p \in R - \{0\}, \alpha \in \mathbf{R}, \lambda > 0)$ is also of the type T_λ, obtained by choosing $G(u) = u^p e^{-u^p}$. Clearly $G \in T(\mathbf{R}^+)$ and, $\Omega(u) = u^{-a} + e^{bu^p} + u^c$, $(a, b, c > 0)$ is a bounding function for G. Indeed, the operators in (i) – (iii) can be obtained from L_λ simply by choosing $p = 1, \alpha = 2$; $p = -1, \alpha = 1$ and $p = -1, \alpha = 0$; respectively. In this general case also $G''(1) \neq 0$.

(v) With the choice $G(u) = \exp\left(-\left(\frac{\log u}{\sigma\sqrt{2}}\right)^2\right)$, $\alpha = 3/2$ and $\lambda = 1/t$, the operators T_λ reduce to the operators $\mathcal{U}_t$ (Micchelli 1969) defined by

$$\mathcal{U}_t(f; x) = \frac{1}{(2\pi\sigma^2 t)^{1/2}} \int_0^\infty f(xe^y) \exp\left[-\frac{1}{2\sigma^2 t}\left(y + \frac{t\sigma^2}{2}\right)^2\right] dy \qquad \text{...(2.8)}$$

which form a semigroup on a certain Banach space of functions on $\mathbf{R}^+$.

Here also $G''(1) \neq 0$ and a bounding function for G is given by

$\Omega(u) = \exp[A(\log u)^2]$. $A > 0$.

3. Basic Approximation

We first prove the following.

Lemma 3.1 — Let $G \in T(\mathbf{R}^+)$ and Ω be a bounding function for G. If

$0 < \delta < a < b < \infty$ and $f \in D_\Omega$, then

$$\lim_{\lambda \to \infty} \lambda^k T_\lambda(f\chi^c_{\delta,x}\,;\, x) = 0, \qquad \ldots(3.1)$$

uniformly in $x \in [a, b]$, for and $k \in \mathbf{R}^+$.

Proof : Since $f \in D_\Omega$, there exist positive constants A, B and M such that $A < \min(1, a)$, $B > \max(b, 1)$ and $|f(u)| \leqslant M\Omega(u)$ for all

$u \in (0, b/B) \cup (a/A, \infty)$. Let $J(A, B) = (0, A) \cup (B, \infty)$.

Then

$$\left.\begin{aligned} &\left| \int_{J(A,B)} u^{\alpha-2} G^\lambda(u) f\left(\frac{x}{u}\right) \chi^c_{\delta,x}\left(\frac{x}{u}\right) du \right| \\ &\leqslant M \int_{J(A,B)} u^{\alpha-2} G^\lambda(u)\, \Omega\left(\frac{x}{u}\right) du. \end{aligned}\right\} \qquad \ldots(3.2)$$

As Ω is a bounding function for G, there exist $\lambda_1, M_1 > 0$ such that $T_{\lambda_1}(\Omega; x) < M_1$ for all $x \in [a, b]$. Using the property (ii) of G, we can find an $\epsilon > 0$ such that $G(u) < G(1) - 2\epsilon$, for almost every $u \in J(A, B)$. Hence, if $\lambda > \lambda_0 > \lambda_1$, we have

$$\left.\begin{aligned} &\int_{J(A,B)} u^{\alpha-2} G^\lambda(u)\, \Omega\left(\frac{x}{u}\right) du \leqslant (G(1)-2\epsilon)^{\lambda-\lambda_0}\, a(\lambda_0)\, T_{\lambda_0}(\Omega; x) \\ &\leqslant M_1 a(\lambda_1)\, [G(1)]^{\lambda_0-\lambda_1}\, (G(1) - 2\epsilon)^{\lambda-\lambda_0} \\ &\qquad = [M(\lambda_0, \lambda_1)/M]\, (G(1) - 2\epsilon)^\lambda, \text{ say.} \end{aligned}\right\} \qquad \ldots(3.3)$$

We can choose a positive δ_1 such that $b/(b + \delta) < 1 - \delta_1 < 1 + \delta_1 < b/(b - \delta)$. In view of the property (ii) of G, if ϵ in the above is chosen sufficiently small, we can also make $G(u) < G(1) - 2\epsilon$, for almost every

$u \in \mathbf{R}^+ - (1 - \delta_1, 1 + \delta_1)$.

Hence

$$\left.\begin{aligned}
&\left| \int_A^B u^{\alpha-2} G^{\lambda}(u)\; f\left(\frac{x}{u}\right) \chi_{\delta,x}\left(\frac{x}{u}\right) du \right| \\
&\leqslant (G(1) - 2\epsilon)^{\lambda-\lambda_0} \int_A^B u^{\alpha-2} G^{\lambda_0}(u) \left| f\left(\frac{x}{u}\right) \right| du \\
&\leqslant (G(1) - 2\epsilon)^{\lambda-\lambda_0} a^{-1} G^{\lambda_0}(1)\,(A^{\alpha} + B^{\alpha}) \int_{a/B}^{b/A} | f(u) |\, du \\
&= (G(1) - 2\epsilon)^{\lambda} M(\lambda_0), \text{ say.}
\end{aligned}\right\} \quad \ldots(3.4)$$

Using the property (i) of G, there exists a $\delta_2 > 0$ such that $G(u) > G(1) - \epsilon$, for all $u \in (1 - \delta_2, 1 + \delta_2)$.

Therefore,

$$a(\lambda) > \int_{1-\delta_2}^{1+\delta_2} u^{\alpha-2} G^{\lambda}(u)\, du > \delta_2 [G(1) - \epsilon]^{\lambda}. \quad \ldots(3.5)$$

Thus (3.2) - (3.5) imply that

$$| T_{\lambda}(f \chi^{c}_{\delta,x};\, x) \leqslant \frac{M(\lambda_0, \lambda_1) + M(\lambda_0)}{\delta_2} \left(\frac{G(1) - 2\epsilon}{G(1) - \epsilon}\right)^{\lambda}$$

since,

$$\lim_{\lambda \to \infty} \lambda^{k} \left(\frac{G(1) - 2\epsilon}{G(1) - \epsilon}\right)^{\lambda} = 0$$

for any $k \in \mathbf{R}^+$, whatsoever the lemma follows.

Next we prove the following basic approximation theorem.

Theorem 3.1 — Let $G \in T(\mathbf{R}^+)$ and Ω be a bounding function for G. If $f \in D_a$ and is continuous at a point $x \in \mathbf{R}^+$, there holds

$$\lim_{\lambda \to \infty} T_{\lambda}(f;\, x) = f(x). \quad \ldots(3.6)$$

Further if f is continuous on $\langle a, b \rangle$, the convergence (3.6) holds uniformly in $x \in [a, b]$.

PROOF : If $f(u)$ is continuous at $u = x$, given an arbitrary $\epsilon > 0$ there exists a $\delta > 0$ such that

$$| f(u) - f(x) | < \epsilon/2,\ | u - x | < \delta \quad \ldots(3.7)$$

where in the uniformity case δ is independent of $x \in [a, b]$. In view of (3.7), for all $u \in \mathbf{R}^+$ there holds

$$| f(u) - f(x) | \leqslant \epsilon/2 + (| f(u) | + | f(x) |) \chi^c_{\delta,x} (u). \qquad \ldots(3.8)$$

Using the linearity, positivity and the property that $T\lambda(1; x) = 1$, of $T\lambda$, from the inequality (3.8) we have

$$| T\lambda(f; x) - f(x) | \leqslant \epsilon/2 + T\lambda((| f(u) | + | f(x) |) \chi^c_{\delta,x} (u); x). \qquad \ldots(3.9)$$

Since $(| f(u) | + | f(x) |) \chi^c_{\delta,x} (u) \in D_\alpha$, using Lemma 3.1, we can find $a\lambda_0$ such that

$$T\lambda((| f(u) | + | f(x) |) \chi^c_{\delta,x} (u); x) < \epsilon/2,$$

for all $\lambda > \lambda_0$ (and $x \in [a, b]$, in the uniformity case).

Hence

$$| T\lambda(f; x) - f(x) | < \epsilon, \lambda > \lambda_0. \qquad (3.10)$$

Since $\epsilon > 0$ is arbitrary, the theorem follows.

4. Voronovskaya Formula

It is intersting to know that under the assumption of the existence of $G'''(1)$ and $G''(1) \neq 0$, the operators $T\lambda$ posses a Voronovskaya-type asymptotic formula. The result is as follows:

Theorem 4.1 — Let $G \in T(\mathbf{R}^+)$ such that $G'(u)$ is absolutely continuous in a certain neighbourhood of $u = 1$, $G'''(1)$ exist and $G''(1)$ be non-zero. Let Ω be a bounding function for G. If $f \in D_\alpha$ and at a certain point $x \in \mathbf{R}^+$, $f''(x)$ exists, then there holds

$$T\lambda(f; x) - f(x) = \frac{xf'(x)\, G(1)\, [2(\alpha - 3)\, G''(1) - G'''(1)]}{2\lambda\, [G''(1)]^2} - \frac{x^2 f''(x)}{2\lambda} \frac{G(1)}{G''(1)} + o\left(\frac{1}{\lambda}\right), (\lambda \to \infty). \qquad \ldots(4.1)$$

Further, if $f''(x)$ exists and is continuous on $\langle a, b \rangle$ then (4.1) holds uniformly in $[a, b]$.

The proof of theorem requires the following auxiliary result.

Lemma 4.1 — Let $G \in T(\mathbf{R}^+)$ such that $G'(u)$ is absolutely continuous in a certain neighbourhood of $u = 1$, $G'''(1)$ exist and $G''(1)$ be non-zero. Then, if $\delta > 0$ is sufficiently small,

(1) $$\lim_{\lambda\to\infty} \frac{\lambda}{a(\lambda)} \int_{1-\delta}^{1+\delta} u^{\alpha-2} G^{\lambda}(u)\, G'(u)\, du = -(\alpha-2)\, G(1),$$

(2) $$\lim_{\lambda\to\infty} \frac{\lambda}{a(\lambda)} \int_{1-\delta}^{1+\delta} u^{\alpha-2} G^{\lambda}(u)\, [G'(u)]^2\, du = -G''(1)\, G(1),$$

and

(3) $$\lim_{\lambda\to\infty} \frac{\lambda}{a(\lambda)} \int_{1-\delta}^{1+\delta} u^{\alpha-2} [G(1) - G(u)]\, G^{\lambda}(u)\, du = \tfrac{1}{2} G(1).$$

PROOF OF THE LEMMA : (1) It is clear that $G'(u)$ exists in a certain neighbourhood of $u = 1$ and is continuous and hence the statement of the lemma is meaningful. Integrating by part

$$\frac{1}{a(\lambda)} \int_{1-\delta}^{1+\delta} u^{\alpha-2} G^{\lambda}(u)\; G'(u)\, du = \frac{1}{a(\lambda)} \left[\frac{u^{\alpha-2} G^{\lambda+1}(u)}{\lambda+1} \right]_{1-\delta}^{1+\delta}$$

$$- \frac{\alpha-2}{\lambda+1} \int_{1-\delta}^{1+\delta} u^{\alpha-3} G^{\lambda}(u)\, du.$$

For a given $\epsilon > 0$, we can choose a $\delta_1 (0 < \delta_1 < \delta)$ such that

$$G(1) - \epsilon \leqslant \frac{G(u)}{u} \leqslant G(1) + \epsilon$$

whenever $u \in (1 - \delta_1, 1 + \delta_1)$. Thus

$$(G(1) - \epsilon) \frac{1}{a(\lambda)} \int_{1-\delta_1}^{1+\delta_1} u^{\alpha-2} G^{\lambda}(u)\, du \leqslant \frac{1}{a(\lambda)} \int_{1-\delta_1}^{1+\delta_1} u^{\alpha-3} G^{\lambda+1}(u)\, du$$

$$\leqslant (G(1) + \epsilon) \frac{1}{a(\lambda)} \int_{1-\delta_1}^{1+\delta_1} u^{\alpha-2} G^{\lambda+1}(u)\, du.$$

Applying Theorem 3.1, we have

$$\lim_{\lambda\to\infty} \frac{1}{a(\lambda)} \int_{1-\delta_1}^{1+\delta_1} u^{\alpha-2} G^{\lambda}(u)\, du = 1.$$

Hence, if λ is sufficiently large, say $\lambda > \lambda_0$,

$$(1-\epsilon) \leqslant \frac{1}{a(\lambda)} \int_{1-\delta_1}^{1+\delta_1} u^{\alpha-2} G^{\lambda}(u)\, du \leqslant (1+\epsilon).$$

Therefore, if $\lambda > \lambda_0$,

$$(1-\epsilon)(G(1)-\epsilon) \leqslant \frac{1}{a(\lambda)} \int_{1-\delta_1}^{1+\delta_1} u^{\alpha-3} G^{\lambda+1}(u)\, du$$
$$\leqslant (1+\epsilon)(G(1)+\epsilon).$$

Let $\| \chi^{c}_{\delta_1,1}\, G \|_\infty = G(1) - 2\mu$. By the property (ii) of G, $\mu > 0$. In view of the property (i), there exists a $\delta_2 (0 < \delta_2 < \delta)$ such that

$$\inf_{|u-1| < \delta_2} G(u) \geqslant G(1) - \mu.$$

Hence, $a(\lambda) \geqslant \delta_2 [G(1)-\mu]^{\lambda}$ and therefore, if λ is sufficiently large

$$\left| \frac{1}{a(\lambda)} \left[\frac{u^{\alpha-2} G^{\lambda+1}(u)}{\lambda+1} \right]_{1-\delta}^{1+\delta} \right| \leqslant \frac{\epsilon}{\lambda}, \quad \lambda > \lambda_1 \quad \text{say.}$$

In view of Theorem 3.1, it is clear that there exists a λ_2 such that

$$\frac{1}{a(\lambda)} \int_{(1-\delta,\,1+\delta)\setminus(1-\delta_1,\,1+\delta_1)} u^{\alpha-2} G^{\lambda}(u)\, \chi^{c}_{\delta_1,1}(u)\, du \leqslant \epsilon, \quad \lambda > \lambda_2, \text{ say.}$$

Making use of the above estimates and the fact that ϵ is arbitrary, we have (1).

(2) The proof uses a similar analysis and the fact that $\lim_{\lambda \to \infty} \frac{a(\lambda+1)}{a(\lambda)} = G(1)$.

Therefore, we leave the proof.

(3) Given an arbitrary $\epsilon > 0$, there exists a $\delta_0 \left(0 < \delta_0 < \frac{1}{1+\delta}\right)$ such that

$$\frac{(1-\epsilon)[G'(u)]^2}{2G''(1)} \leqslant G(u) - G(1) \leqslant \frac{(1+\epsilon)[G'(u)]^2}{2G''(1)}, \; | u^{-1} - 1 | < \delta_0$$

Now, using the arguments given in the proof of part 1, the proof easily follows.

This completes the proof of the lemma.

Proof of Theorem: The properties of G imply that $G'(1) = 0$. Using l' Hospital's rule, we have

$$\lim_{u \to 1} \frac{f\left(\frac{x}{u}\right) - f(x) + \frac{xf'(x)}{G''(1)}\left[G'(u) - \frac{G'''(1)}{G''(1)} G(u) + \frac{G(1)G'''(1)}{G''(1)}\right]}{G(1) - G(u)} \times$$

(*equation continued on p.* 21)

$$\times \frac{-[2xf'(x)+x^2f''(x)]\left[\frac{G(u)-G(1)}{G''(1)}\right]}{G(1)-G(u)} = 0.$$

Hence, given an arbitrary $\epsilon > 0$ there exists a $\delta > 0$ such that if u satisfies $\left|\frac{x}{u} - x\right| < \delta$, there holds

$$\left| f\left(\frac{x}{u}\right) - f(x) + \frac{xf'(x)}{G''(1)}\left[G'(u)\frac{G'''(1)}{G''(1)}G(u) + \frac{G(1)\,G''(1)}{G''(1)}\right] - [2xf'(x)+x^2f''(x)]\left[\frac{G(u)-G(1)}{G''(1)}\right]\right| \leqslant \epsilon\,[G(1)-G(u)].$$

Moreover, it is easily seen that in the uniformity case the above δ can be chosen independently of $x \in [a, b]$. Multiplying this inequality by $u^{\alpha-2}G^{\lambda}(u)/a(\lambda)$, integrating between the limits $(1-\delta, 1+\delta)$ and making use of Lemma 4.1, we have

$$-\frac{\epsilon}{2}G(1) \leqslant \varlimsup_{\lambda\to\infty}\frac{\lambda}{a(\lambda)}\int_{1-\delta}^{1+\delta}\left[f\left(\frac{x}{u}\right)-f(x)\right]u^{\alpha-2}G^{\lambda}(u)\,du - \frac{xf'(x)}{G''(1)}(\alpha-2)\,(G(1)) + \frac{xf'(x)\,G'''(1)\,G(1)}{2(G''(1))^2} + \frac{[2xf'(x)+x^2f''(x)]\,G(1)}{2G''(1)} \leqslant \frac{\epsilon}{2}G(1)$$

$$\lim_{\lambda\to\infty}\frac{\lambda}{a(\lambda)}\int_{(0,\infty)-(1-\delta,1+\delta)}\left[f\left(\frac{x}{u}\right)-f(x)\right]u^{\alpha-2}G^{\lambda}(u)\,du = 0,$$

which holds uniformly in $x \in [a, b]$, in the uniformity case. Hence

$$-\frac{\epsilon}{2}G(1) \leqslant \varlimsup_{\lambda\to\infty}[T_{\lambda}(f;x)-f(x)] - \frac{(\alpha-2)\,G(1)}{G''(1)}xf'(x) + xf'(x)\frac{G'''(1)\,G(1)}{2(G''(1))^2} + \frac{[2xf'(x)+x^2f''(x)]\,G(1)}{2G''(1)} \leqslant \frac{\epsilon}{2}G(1).$$

In view of the fact that $\epsilon > 0$ arbitrary, the result follows.

Corollary 4.1 — Taking $G(u) = ue^{-u}$, $\alpha = 2$ and $\lambda = n$ we obtain the Voronovskaya formula for the Gamma operators of Müller

$$G_n(f;x) - f(x) = \frac{x^2 f''(x)}{2n} + o(n^{-1}) \quad (n \to \infty).$$

Corollary 4.2 — With $G(u) = u^{-1}e^{-u^{-1}}$, $\alpha = 1$ and $\lambda = n$ we have the Voronovskaya formula for the operators S_n^1

$$S_n^1\,(f;\,x) - f(x) = \frac{x^2 f''(x)}{2n} + o\,(n^{-1}),\ (n \to \infty).$$

Corollary 4.3 — Choosing $G(u)$ as in the Corollary 4.2, $\alpha = 0$ and $\lambda = k$, we have the Voronovskaya formula for the operators $L_{k,t}$.

$$L_{k,t}[f;\,x] = \frac{x f'(x)}{k} + \frac{x^2 f''(x)}{2k} + o\,(k^{-1})\,(k \to \infty).$$

Corollary 4.4 — If we take $G(u) = u^p e^{-u^p}$ and $p\ (\neq 0)$, α any real numbers then we find the Voronovskaya formula for the operators L_λ as follows.

$$L_\lambda(f;\,x) - f(x) = \frac{x f'(x)}{2\lambda p^2}\,(p - 2\alpha + 3) + \frac{x^2 f''(x)}{2\lambda p^2} + o\,(\lambda^{-1}),\ (\lambda \to \infty).$$

Corollary 4.5 — Taking $G(u) = \exp\left[-\left(\frac{\log u}{\sqrt{2\sigma}}\right)^2\right]$, $\alpha = 3/2$ and $\lambda = 1/t$, we obtain the Voronovskaya formula for the operators $\mathcal{U}_t$ (Micchelli 1969)

$$\mathcal{U}_t(f;\,x) - f\,(x) = \frac{x^2 f''(x)\,\sigma t}{2\sqrt{2}} + o\,(t)\,(t \to 0^+).$$

5. Effor Estimates

In the previous section we obtained a precise formula giving the rate of convergence of $T_\lambda(f;\,x)$ to $f(x)$. The assumption on f has been the existence of its second derivative. If f is only assumed to be continuous, the following theorem gives an estimate of error $|\,T_\lambda(f;\,x) - f(x)\,|$ in terms of the modulus of continuity of f.

Theorem 5.1 — Let $G \in (\mathbf{R}^+)$, $G'''(1)$ exist and $G''(1)$ be non-zero. Then there holds

$$|\,T_\lambda(f;\,x) - f(x)\,| \leqslant \omega_f(\lambda^{-1/2})\left[1 + \min\left(x^2\left\{-\frac{G(1)}{G''(1)} + o\,(1)\right\},\ x\sqrt{\left\{-\frac{G(1)}{G''(1)} + o\,(1)\right\}}\right)\right],\ x \in \mathbf{R}^+,\ (\lambda \to \infty) \quad \ldots(5.1)$$

where ω_f denotes the modulus of continuity of f and the $o\,(1)$ are independent of x.

Proof : Using (4.1), we have

$$T_\lambda((u - x)^2;\,x) = x^2\left[-\frac{G(1)}{\lambda G''(1)} + o\,(\lambda^{-1})\right],\ (\lambda \to \infty). \quad \ldots(5.2)$$

By elementary properties of modulus of continuity

$$| f(u) - f(x) | \leqslant \omega_f(\lambda^{-1/2}) [1 + \lambda^{1/2} | u - x |] \qquad ...(5.3)$$

and also

$$| f(u) - f(x) | \leqslant \omega_f(\lambda^{-1/2}) [1 + \lambda (u - x)^2] \qquad ...(5.4)$$

for all $x, u \in \mathbf{R}^+$. By Schwarz's inequality (5.2) implies

$$T_\lambda(| u - x | ; x) \leqslant \frac{x}{\sqrt{\lambda}} \sqrt{\left\{- \frac{G(1)}{G''(1)} + o(1)\right\}}, \ (\lambda \to \infty). \qquad ...(5.5)$$

Making use of the linearity and positivity of the operators T_λ, (5.1) follows from (5.2) - (5.5).

Remark : The proof of (5.1) is based only on the evaluation (5.2), which we have deduced from Theorem 4.1. One can prove (Kunwar 1979, Chap. II) that (5.2) is valid if only it is assumed that $G''(1)$ exists and is non-zero.

For functions which are continuously differentiable the error estimate (5.1) is rather conservative and a better estimate is as follows:

Theorem 5.2 — Let $G \in T(\mathbf{R}^+)$, $G'''(1)$ exist and $G''(1)$ be non-zero. Then if f' exists and is uniformly continuous of $\mathbf{R}^+$ there holds

$$| T_\lambda(f; x) - f(x) | \leqslant \frac{x \, | f'(x) |}{(G''(1))^2} [2(\alpha - 3) G''(1) - G'''(1) + o(1)]$$

$$+ \omega_{f'}(\lambda^{-1/2}) \left[\frac{x}{\sqrt{\lambda}} \left(\left\{ - \frac{G(1)}{G''(1)} \right\}^{1/2} + o(1) \right) \right.$$

$$+ \frac{x^2}{2\sqrt{\lambda}} \left(\left\{ - \frac{G(1)}{G''(1)} \right\} \right.$$

$$\left. \left. + o(1) \right) \right] (x \in \mathbf{R}^+, \ \lambda \to \infty), \qquad ...(5.6)$$

where $\omega_{f'}$, denotes the modulus of continuity of f'.

PROOF : We have

$$| f(u) - f(x) - (u - x) f'(x) | \leqslant | \int_x^u f'(u) - f'(x) \, du |$$

$$\leqslant | \int_x^u \omega_{f'}(| u - x |) \, du | \leqslant \int_x^u \omega_{f'}(\lambda^{-1/2}) (1 + \lambda^{1/2} | u - x |) \, du |$$

$$= \omega_{f'}(\lambda^{-1/2}) \{ | u - x | + \tfrac{1}{2} \lambda^{1/2} (u - x)^2 \}. \qquad ...(5.7)$$

Since, by Theorem 4.1 we have

$$T_\lambda((u-x);x) = \frac{x}{2\lambda}\,\frac{G(1)\,[2(\alpha-3)\,G''(1) - G'''(1)]}{[G''(1)]^2} + o\,(\lambda^{-1}) \quad \ldots(5.8)$$

the inequality (5.6) follows by operating (5.7) by T_λ and making use of (5.2), (5.4) and (5.8).

Remark : Notice that in Theorems 5.1 and 5.2, no explicit assumption like $f \in D_\Omega$ is made. Indeed, since the uniform continuity of f or f' implies that $f(u) = O\,(u^2)$, $u \to \infty$, it follows that $f \in D_\Omega$ with $\Omega(u) = 1 + u^2$ which is always a bounding function for each $G \in T(\mathrm{R}^+)$.

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