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ON THE LOWER BOUNDS OF THE NUMBER OF REAL ROOTS OF A RANDOM ALGEBRAIC EQUATION

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The lower bound of the number of real roots of the family of equations

$$f_n(x, t) = \sum_{k=0}^{n} a_k(t)x^k = 0$$

where $a_k(t)$, $0 \leqslant t \leqslant 1$, are dependent random variables assuming real values only and following normal distribution with mean zero and joint density function

$$| M |^{1/2}(2\pi)^{-n/2} \exp[-\tfrac{1}{2}\bar{a}'M\bar{a}]$$

where M^{-1} is the moment matrix with $\sigma_i = 1$, $\rho_{ij} = \rho$, $0 < \rho < 1$, $i \neq j$, $i, j = 0, 1, 2, \ldots, n$ and $\bar{a}'$ is the transpose of the column vector $\bar{a}$, is estimated.

§1. Consider the family of equations

$$f_n(x, t) = \sum_{k=0}^{n} a_k(t)\, x^k = 0 \qquad \ldots(1.1)$$

where $a_k(t)$, $0 \leqslant t \leqslant 1$, are dependent random variables assuming real values only and following normal distribution with mean zero and joint density function

$$| M |^{1/2}(2\pi)^{-n/2} \exp[-\tfrac{1}{2}\bar{a}'M\bar{a}] \qquad \ldots(1.2)$$

where M^{-1} is the moment matrix with $\sigma_i = 1$, $\rho_{ij} = \rho$, $0 < \rho < 1$, $i \neq j$, $i, j = 0, 1, \ldots, n$ and $\bar{a}'$ is the transpose of the column vector $\bar{a}$.

Here we estimate the lower bound of the number of real roots of (1.1).

Theorem — There exists an integer n_0 and a set E of measure atmost $\frac{B \log\log n_0}{\log n_0}$ such that for each $n > n_0$ and all t not belonging to E, eqns. (1.1) have atleast $\frac{\beta \log n}{\log\log n}$ roots where β and B are constants.

The transformation $x \to 1/x$ makes the equation $f_n(x, t) = 0$ transformed to $\sum_{r=0}^{n} a_{n-r}(t)\, x^r = 0$ and $(a_0(t), a_1(t), \ldots, a_n(t))$ and $(a_n(t), \ldots, a_0(t))$ have the same joint density function. Therefore the number of roots and the measure of the exceptional set in the range $[-\infty, \infty]$ are twice the corresponding estimates of the range $[-1, 1]$. We consider the range $[0, 1]$ only and using the same procedure $[-1, 0]$ can be considered and show that this lower bound is same as in $[0, 1]$. Thus the number of roots in the range $[-\infty, \infty]$ and the measure of the exceptional set are each 4 times the corresponding estimates for the range $[0, 1]$. Evans (1965) has considered the case when the random coefficients are independent and normal. Sambandham (1979) has considered the case for the upper bound of the number of real roots of a random algebraic equation, with dependent random variables. The effect of ρ makes the bounds of the real roots of the equation, narrower than that of the independence case.

§2. We write

$$f_n(x, t) = \sum_{k=0}^{n} a_k(t)\, x^k$$

$$= \sum_{k=0}^{p_m} a_k(t)\, x^k + \sum_{k=p_m+1}^{q_m} a_k(t)\, x^k + \sum_{k=q_m+1}^{n} a_k(t)\, x^k$$

where $p_m = (m-1)! \log m$ and $q_m = m! \log m$.

Let us define $x_m = 1 - \dfrac{1}{m!}$ and $\dfrac{3x_m - 1}{2} = \alpha$ and $\dfrac{x_m + 1}{2} = \beta$. We define the random variables

$$\eta_m = 1 \text{ if } \sum_{p_m+1}^{q_m} a_k \alpha^k \text{ and } \sum_{p_m+1}^{q_m} a_k \beta^k \text{ are of opposite sign and}$$

$$\Big|\sum_{p_m+1}^{q_m} a_k \alpha^k\Big| > \delta \text{ and } \Big|\sum_{p_m+1}^{q_m} a_k \beta^k\Big| > \delta$$

$$= 0, \text{ otherwise.}$$

$$\zeta_m = 1 \text{ if either } \Big|\sum_{0}^{p_m} a_k \alpha^k\Big| > \delta/2 \text{ or } \Big|\sum_{0}^{p_m} a_k \beta^k\Big| > \delta/2$$

$$\text{or } \Big|\sum_{q_m+1}^{n} a_k \alpha^k\Big| > \delta/2 \text{ or } \Big|\sum_{q_m+1}^{n} a_k \beta^k\Big| > \delta/2$$

$$= 0, \text{ otherwise}$$

and the sets

$$E_1 = \{t : \Big|\sum_{0}^{p_m} a_k \alpha^k\Big| > \delta/2 \text{ or } \Big|\sum_{0}^{p_m} a_k \beta^k\Big| > \delta/2\}$$

$$E_2 = \{t : \sum_{p_m+1}^{q_m} a_k\alpha^k \text{ and } \sum_{p_m+1}^{q_m} a_k\beta^k \text{ are of opposite signs}$$

and $$\Big| \sum_{p_m+1}^{q_m} a_k\alpha^k \Big| > \delta \text{ and } \Big| \sum_{p_m+1}^{q_m} a_k\beta^k \Big| > \delta\}$$

$$E_3 = \{t : \Big| \sum_{q_m+1}^{n} a_k\alpha^k \Big| > \delta/2 \text{ or } \Big| \sum_{q_m+1}^{n} a_k\beta^k \Big| > \delta/2\}.$$

Note that $\{t : \eta_m = 1\} = E_2$; $\{t : \zeta_m = 1\} = E_1 \cup E_3$.

Defining $\xi_m = \eta_m - \eta_m\zeta_m$.

We find that the probability that $f_n(x, t)$ has a zero in (α, β) is not less than $m(t : \xi_m = 1)$.

The probability distribution function of the random variable $\sum_{v=0}^{n} a_v(t)g_v$ is given by

$$\frac{1}{\sqrt{2\pi}\sigma_n} \int_{-\infty}^{x} \exp\left(-\frac{u^2}{2\sigma_n^2}\right) du$$

where $$\sigma_n^2 = (1-\rho) \sum_{v=0}^{n} g_v^2 + \rho\Big(\sum_{v=0}^{n} g_v\Big)^2$$

Let $$\sigma^2 = (1-\rho) \sum_{v=0}^{\infty} g_v^2 + \rho\Big(\sum_{v=0}^{\infty} g_v\Big)^2, \qquad \sigma \geqslant \sigma_n.$$

$$F = \{ \Big| \sum_{0}^{n} a_v(t)\, g_v \Big| \geqslant \lambda\sigma\}$$ Let $g_v = b_v + ic_v$ where b_v and c_v are real

$$G = \left\{t : \left| \sum_{v=0}^{n} a_v(t)\, b_v \right| \geqslant \frac{\lambda\sigma}{\sqrt{2}}\right\}$$

$$H = \left\{t : \left| \sum_{v=0}^{n} a_v(t)\, c_v \right| \geqslant \frac{\lambda\sigma}{\sqrt{2}}\right\}$$

$$m(G) = \sigma_n^{-1} \sqrt{2/\pi} \int_{\lambda\sigma_n/\sqrt{2}}^{\infty} \exp\left(-\frac{u^2}{2\sigma_n^2}\right) du$$
$$\leqslant \left(\frac{\sigma_n}{\lambda\sigma}\right) \sqrt{2/\pi}\; \exp\left(\frac{-\lambda^2\sigma^2}{4\sigma_n^2}\right)$$

(*equation continued on p.* 151)

$$\leqslant \frac{2}{\sqrt{\pi}} \frac{1}{\lambda} \exp\left(\frac{-\lambda^2}{4}\right)$$

$$m(H) < \frac{2}{\sqrt{\pi}} \frac{1}{\lambda} \exp\left(\frac{-\lambda^2}{4}\right)$$

$$F \subset G \cup H \Rightarrow m(F) \leqslant m(G) + m(H)$$

$$\leqslant \frac{4}{\lambda\sqrt{\pi}} \exp\left(\frac{-\lambda^2}{4}\right). \qquad \ldots(2.1)$$

§3. In this section we calculate $m(E_1 \cup E_3)$. First we calculate $m(E_1)$. Let us define the sets

$$G_1 = \left\{ t : \left| \sum_0^{p_m} a_k(t)\, \alpha^k \right| > \lambda \sqrt{(1-\rho) \sum_0^{p_m} \beta^{2k} + \rho\left(\sum_0^{p_m} \beta^{2k}\right)^2} \right\}$$

$$\subseteq \left\{ t : \left| \sum_0^{p_m} a_k(t)\, \alpha^k \right| > \lambda \sqrt{1-\rho} \sqrt{\sum_0^{p_m} \beta^{2k}} \right\}$$

and

$$G_2 = \left\{ t : \left| \sum_0^{p_m} a_k(t)\, \beta^k \right| > \lambda \sqrt{1-\rho} \sqrt{\sum_0^{p_m} \beta^{2k}} \right\}$$

$$\Rightarrow m(G_1) \leqslant m(G_2)$$

By (2.1)

$$m(G_2) \leqslant \frac{4}{\lambda\sqrt{\pi}\sqrt{1-\rho}} \exp\left(-(1-\rho)\frac{\lambda^2}{4}\right)$$

Now

$$\sum_0^{p_m} \beta^{2k} < p_m + 1 < (m-1)!\,(\log m) + 1.$$

Let us take $\lambda = \sqrt{\left(\frac{m}{\log m}\right)}$ and define the sets

$$G_1' = \{t : \left| \sum_0^{p_m} a_k(t)\, \alpha^k \right| > d\sqrt{m!}\}$$

$$G_2' = \{t : \left| \sum_0^{p_m} a_k(t)\, \beta^k \right| > d\sqrt{m!}\}$$

where

$$d^2 > 1 + \frac{1}{\log 2}.$$

Since $m(G_1') \leqslant m(G_1)$ and $m(G_2') \leqslant m(G_2)$,

$$m(G_1') \leqslant m(G_2') < \frac{4}{\sqrt{1-\rho}\sqrt{\pi}} \sqrt{\frac{\log m}{m}} \exp\left(-m(1-\rho)/4 \log m\right).$$

Then if $\frac{1}{2}\delta \geqslant d\sqrt{m!}$ we have $E_1 \subseteq G_1' \cup G_2'$ and $m(E_1) \leqslant m(G_1') + m(G_2')$

$$\therefore m(E_1) \leqslant 2\left(\frac{4}{\sqrt{1-\rho}\sqrt{\pi}}\right)\sqrt{\frac{\log m}{m}}\exp(-m(1-\rho)/4\log m).$$

Next we calculate $m(E_3)$. Let us define the set

$$H_1 = \left\{\{t : \left|\sum_{q_m+1}^{n} a_k(t)\,\alpha^k\right| > \mu\sqrt{1-\rho}\sqrt{\sum_{q_m+1}^{n} \beta^{2v}}\right\}$$

$$H_2 = \left\{t : \left|\sum_{q_m+1}^{n} a_k(t)\,\beta^k\right| > \mu\sqrt{1-\rho}\sqrt{\sum_{q_m+1}^{n} \beta^{2v}}\right\}$$

so that $m(H_1) \leqslant m(H_2)$.

By (2.1) $$m(H_2) < \frac{4}{\mu\sqrt{\pi}\sqrt{1-\rho}}\exp(-\mu^2(1-\rho)/4)$$

$$\sum_{q_m+1}^{n} \beta^{2k} = \sum_{q_m+1}^{n}\left(1-\frac{1}{2m!}\right)^{2k} < \sum_{q_m+1}^{n}\exp(-k/m!)$$

$$= \frac{\exp(-(q_m+1)/m!)}{1-\exp(-1/m!)} < \frac{m!\exp(-\log((m-1)/m!))}{1-(1/2m!)}$$

$$< \tfrac{4}{3}(m-1)! \text{ for } m \geqslant 2.$$

Let us take $\mu = \sqrt{\dfrac{m}{\log m}}$ and define the sets

$$H_1' = \left\{t : \left|\sum_{q_m+1}^{n} a_k(t)\,\alpha^k\right| > \sqrt{1-\rho}\sqrt{\left(\frac{4}{3}\frac{m!}{\log m}\right)}\right\}$$

$$H_2' = \left\{t : \left|\sum_{q_m+1}^{n} a_k(t)\,\beta^k\right| > \sqrt{1-\rho}\sqrt{\left(\frac{4}{3}\frac{m!}{\log m}\right)}\right\}.$$

Since $m(H_1') \leqslant m(H_1)$ and $m(H_2') \leqslant m(H_2)$

$$m(H_1') \leqslant m(H_2') < \frac{4}{\sqrt{\pi}\sqrt{1-\rho}}\sqrt{\frac{\log m}{m}}\exp(-(1-\rho)\,m/4\log m).$$

Then if $\dfrac{\delta}{2} \geqslant \sqrt{\dfrac{4}{3}\dfrac{m!}{\log m}}$

We have

$$E_3 \subseteq H_1' \cup H_2' \text{ and } m(E_3) \leqslant m(H_1') + m(H_2')$$

$$m(E_3) < 2 \cdot \frac{4}{\sqrt{\pi}} \sqrt{\left(\frac{\log m}{m}\right)} \frac{1}{\sqrt{1-\rho}} \exp\left(-(1-\rho)m/4 \log m\right).$$

§4. In this section we calculate $m(E_2)$. We want to find the probability density function of the two dimensional random variable

$$\left(\sum_{k=pm+1}^{qm} a_k(t)\, \alpha^k, \sum_{k=pm+1}^{qm} a_k(t)\, \beta^k \right).$$

The probability density function is $P(e)$ where

$$P(e) = \begin{cases} 0 \text{ unless } e \text{ intersects the line } \dfrac{u}{\alpha^k} = \dfrac{v}{\beta^k} = \theta \\ \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{\theta_1}^{\theta_2} \exp(-\theta^2/2)\, d\theta \text{ if } e \text{ intersects } \dfrac{u}{\alpha^k} = \dfrac{v}{\beta^k} = \theta \\ \qquad \text{in the segment } (\theta_1, \theta_2). \end{cases}$$

The 2-dimensional random variable $\left(\sum_{pm+1}^{qm} a_k\alpha^k, \sum_{pm+1}^{qm} a_k\beta^k \right)$ is Gaussian, centred, with covariance matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

where

$$a = (1-\rho) \sum_{pm+1}^{qm} \alpha^{2v} + \rho\left(\sum_{pm+1}^{qm} \alpha^v \right)^2$$

$$b = (1-\rho) \sum_{pm+1}^{qm} \alpha^v\beta^v + \rho \sum_{pm+1}^{qm} \alpha^v \sum_{pm+1}^{qm} \beta^v$$

$$c = (1-\rho) \sum_{pm+1}^{qm} \beta^{2v} + \rho\left(\sum_{pm+1}^{qm} \beta^v \right)^2.$$

Now $b^2 < ac$.

The probability density function is

$$\frac{1}{2\pi\sqrt{ac-b^2}} \exp\left(-av^2 - 2buv + cu^2/2(ac-b^2)\right).$$

Hence we have

$$m(E_2) = \frac{1}{2\pi\sqrt{ac-b^2}} \left\{ \int_{-\infty}^{-\delta} du \int_{+\delta}^{\infty} dv + \int_{\delta}^{\infty} du \int_{-\infty}^{-\delta} dv \right\}$$
$$\times \exp\left(-av^2 - 2buv + cu^2\right)/2(ac-b^2)$$

(*equation continued on p.* 154)

$$= \frac{1}{\pi\sqrt{ac-b^2}} \int_{\delta}^{\infty}\int \exp(-av^2 - 2buv + cu^2)/2(ac - b^2)\, du\, dv$$

$$= \frac{1}{\pi} \int_{\delta/\sqrt{a}}^{\infty} \exp(-s^2/2)\, ds \int_{(\delta - ba^{-1/2}s)\, a^{1/2}/(ac-b^2)}^{\infty} \exp(-t^2/2)\, dt$$

Now $$a = (1-\rho) \sum_{p_m+1}^{q_m} \alpha^{2v} + \rho \left(\sum_{p_m+1}^{q_m} \alpha^{v} \right)^2$$

$$a \sim \rho_9^4 (m!)^2 \alpha^{2(p_m+1)};\ b \sim \tfrac{4}{3}\rho(m!)^2(\alpha\beta)^{p_m+1};\ c \sim 4\rho(m!)^2\beta^{2(p_m+1)}$$

$$m(E_2) = \frac{1}{\pi} \int_{\delta/\sqrt{a}}^{\infty} \exp(-s^2/2)\, ds \int_{(\delta - ba^{-1/2}s)\, a^{1/2}/(ac-b^2)}^{\infty} \exp(-t^2/2)\, dt$$

$$\geqslant \frac{1}{\pi} \int_{\delta/\sqrt{a}}^{\infty} \exp(-s^2/2)\, ds \int_{0}^{\infty} \exp(-t^2/2)\, dt \text{ since } \left(\delta - \frac{b}{\sqrt{a}}\right) < 0$$

$$\geqslant \frac{\sqrt{\pi}}{2\pi} \int_{\delta/\sqrt{(1-\rho)\Sigma\alpha^{2v}}}^{\infty} \exp(-s^2/2)\, ds$$

$$\geqslant \frac{1}{2\sqrt{\pi}} \int_{\delta/\sqrt{(1-\rho)}\sqrt{\Sigma\alpha^{2v}}}^{\infty} \exp(-s^2/2)\, ds$$

$> L$ where L is a constant (since $\dfrac{\delta}{\sqrt{1-\rho}\ \sqrt{\Sigma\alpha^{2v}}} < \dfrac{2\sqrt{3}\ de}{\sqrt{(1-\rho)}}$.

§5. In this section we find a lower bound for N_n, the number of zeros of $f_n(x, t)$ in [0, 1]. We require the following version of the strong law of large numbers.

Theorem A — If $\eta_2, \eta_3, \ldots$ are independent random variables with $v(\eta_i) < 1$, all i, then given any $\epsilon > 0$, we have

$$Pr\left\{ \sup_{k \geqslant k_0+1} \left| \frac{1}{k-1} \sum_{2}^{k} (\eta_v - E(\eta_v)) \right| < \epsilon \right\} \geqslant 1 - \frac{16}{\epsilon^2 k_0}.$$

From the definition of random variable ξ_m in §2 it is clear that $N_n \geqslant \sum_{m=2}^{k} \xi_m$.

Since $\xi_m = \eta_m - \eta_m\zeta_m$, we have

$$\sum_2^k (\xi_m - E(\eta_m)) = \sum_2^k \{(\eta_m - \eta_m\zeta_m) - E(\eta_m)\}$$

$$= \sum_2^k (\eta_m - E(\eta_m) - \sum_2^k \eta_m\zeta_m$$

$$\Rightarrow \left| \sum_2^k (\xi_m - E(\eta_m)) \right| \leqslant \left| \sum_2^k (\eta_m - E(\eta_m)) \right| + \left| \sum_2^k \eta_m\zeta_m \right|$$

Since $|\eta_m| \leqslant 1$,

$$\left| \frac{1}{k-1} \sum_2^k (\xi_m - E(\eta_m)) \right|$$

$$\leqslant \left| \frac{1}{k-1} \sum_2^k (\eta_m - E(\eta_m)) \right| + \left| \frac{1}{k-1} \sum_2^k \zeta_m \right|$$

$$E\left(\frac{1}{k-1} \sum_2^k \zeta_m\right) = \frac{1}{k-1} \sum_2^k \int \zeta_m \, dt$$

$$\leqslant \frac{1}{k-1} \sum_2^k \frac{16}{\sqrt{\pi}} \sqrt{\frac{\log m}{m(1-\rho)}} \exp(-(1-\rho)m/4 \log m).$$

Hence $\dfrac{1}{k-1} \displaystyle\sum_2^k \zeta_m < \epsilon_1$ outside an exceptional set of measure atmost $\dfrac{16}{\epsilon_1(k-1)}$

$\dfrac{1}{\sqrt{\pi(1-\rho)}} \displaystyle\sum_2^k \sqrt{\frac{\log m}{m}} \exp(-(1-\rho)m/4 \log m) = \psi_k$ (say) put $k_v = 2^v$ where v is an integer. Then

$$\operatorname*{Sup}_{v \geqslant v_0} \frac{1}{k_v - 1} \sum_2^{k_v} \zeta_m < \epsilon_1$$

outside a set of measure atmost $\displaystyle\sum_{v=v_0}^{\infty} \psi_{2^v}$

If $k_v < k < k_{v+1}$ then $\dfrac{1}{k-1} \displaystyle\sum_2^k \zeta_m < \dfrac{3}{k_{v+1}-1} \sum_2^{k_{v+1}} \zeta m$ and putting $k_0 = 2^{v_0}$

we have $\operatorname*{Sup}_{k \geqslant k_0} (k-1)^{-1} \sum_2^k \zeta_m < 3\epsilon_1$ outside a set of measure atmost

$$\sum_{\nu=\nu_0}^{\infty} \frac{Q}{\epsilon_1(2^\nu - 1)} < \frac{Q}{\epsilon_1 2^{\nu_0 - 2}} = \frac{4Q}{\epsilon_1 . k_0}$$

where $$Q = \frac{16}{\sqrt{\pi(1-\rho)}} \sum_{m=2}^{\infty} \sqrt{\frac{\log m}{m}} \exp\left(-(1-\rho)\, m/4 \log m\right).$$

Thus outside this exceptional set, we have

$$\operatorname*{Sup}_{k \geqslant k_0} \left| \frac{1}{k-1} \sum_2^k (\xi_m - E(\eta_m)) \right| \leqslant \operatorname*{Sup}_{k \geqslant k_0} \left| \frac{1}{k-1} \sum_2^k (\eta_m - E(\eta_m)) \right| + 3\epsilon_1.$$

Hence $$Pr\left\{ \operatorname*{Sup}_{k \geqslant k_0} \left| \frac{1}{k-1} \sum_2^k (\xi_m - E(\eta_m)) \right| < \epsilon \right\}$$

$$\geqslant Pr\left\{ \operatorname*{Sup}_{k \geqslant k_0} \left| \frac{1}{k-1} \sum_2^k (\eta_m - E(\eta_m)) + 3\epsilon_1 < \epsilon \right\}$$

$$\geqslant Pr\left\{ \operatorname*{Sup}_{k \geqslant k_0} \left| \frac{1}{k-2} \sum_2^k (\eta_m - E(\eta_m)) < \epsilon - 3\epsilon_1 \right\}$$

$$\geqslant 1 - \frac{16}{(\epsilon - 3\epsilon_1)^2 k_0} \text{ by Theorem A.}$$

Then outside a set X_{k_0} where $m(X_{k_0}) \leqslant \frac{16}{(\epsilon - 3\epsilon_1)^2 k_0} + \frac{4Q}{\epsilon_1 . k_0}$ we have $(k-1)^{-1} \sum_2^k \xi_m > (k-1)^{-1} \sum_2^k E(\eta_m) - \epsilon$ for all $k \geqslant k_0$. Hence outside X_{k_0}, $\sum_2^k \xi_m > \sum_2^k E(\eta_m) - (k-1)\,\epsilon > (k-1)(L-\epsilon)$ since $E(\eta_m) > L$ for all m. Choose $\epsilon = \frac{1}{2} L$ and $\epsilon_1 = \frac{1}{8} L$. Then outside X_{k_0} where $m(X_{k_0}) < k_0^{-1}(32^2 L^{-2} + 32 Q L^{-1}$. We have $\sum_2^k \xi_m > \frac{1}{2}(k-1)\, L$, for every $k \geqslant k_0$ choose k to satisfy the inequality, $k! \log k < n < k(k!) \log k$. Then $N_n > \sum_2^k \xi_m > \frac{L}{2}\left(\frac{\log n}{\log \log n} - 1\right)$ for every $k \geqslant k_0$ i.e. for every $n > n_0$, outside an exceptional set of measure atmost $\frac{\log \log n_0}{\log n_0}\left(\frac{32^2}{L^2} + \frac{32 Q}{L}\right)$.

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