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ON THE GLOBAL CONTROLLABILITY OF A CERTAIN CLASS OF MINIMUM TIME CONTROL PROBLEMS

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There exists a large class of minimum time control problems relating to linear systems which cannot be tackled by Pontryagin's maximum principle by reason of the nature of the constraints on the control functions. Many such problems can be looked upon as a linear transformation from one normed linear space to another and the constraint can be expressed as one on the norm. The solution of the minimum time control problem, can then be obtained from that of the auxiliary problem of minimisation of the norms for a terminal time given in advance. Chaudhuri and Mukherjee (1981) have discussed how functional analytic techniques can be used to solve such problems. In this paper certain necessary and sufficient conditions for global controllability of such problems has been proved.

Here we shall define as in Chaudhuri and Mukherjee (1981) the time optimal control problem as follows : Let B_t be a Banach space depending upon the continuous parameter t. Let D be another Banach space. Let T_t be a transformation depending upon the parameter t, mapping B_t onto D. Let $U_t \subset B_t$ be the unit ball in B_t and $\xi \in D$. The problem is to determine $u \in U_t$ such that $T_t\, u = \xi$ and t is minimum. Here we shall consider only the case when T_t is linear, bounded and onto.

In the problems which usually arise in practice, B_t is an increasing function of t in the sense that $B_{t_1} \subset B_{t_2}$ whenever $t_1 \leqslant t_2$. Also T_{t_1} can be regarded as the restriction of T_{t_2} defined on B_{t_2}, on B_{t_1}. It is easy to show that under the above conditions $U_{t_1} \subset U_{t_2}$.

The necessary and sufficient conditions for controllability has been discussed by many authors (see References) for the minimum time problem in the special case when B_t coincides with $L_\infty(0, t)$ and D is n-dimensional Euclidean space. The necessary and sufficient condition for controllability for more general Banach spaces will be discussed here. For the sake of completeness, we shall state the following definitions and theorems already proved in Chaudhuri and Mukherjee (1981).

Reachable Region (*Set*)

The set of all points $\xi \in D$ such that $T_t u = \xi$ for some $u \in U_t$ will be called the Reachable Region (set) with respect to the linear transformation T_t and will be denoted by $C(t)$.

For brevity, we shall suppress the dependence of B and T on t.

Theorem 1 — The reachable region $C(t)$ is bounded and a convex body, symmetrical with respect to the origin of D.

PROOF : Since U is convex and bounded and T is linear and bounded, and since linear operators preserve convexity, the image $C(t) = TU$ is convex and bounded. Also, if λ is any scalar with $|\lambda| \leqslant 1$ then for $\xi \in C(t)$ we also have $\lambda\xi \in C(t)$ because $\xi = Tu$ for some $u \in U$ implies $\lambda\xi = \lambda Tu = T(\lambda u)$ and $\lambda u \in U$ due to $\|\lambda u\| = |\lambda| \, \|u\| \leqslant 1$. Thus, $C(t)$ is circled and symmetrical. Again, since T is onto, it follows from open mapping theorem that $TU = C(t)$ will contain a multiple of the unit ball in D. Thus, $C(t)$ is a convex set with nonempty interior and thus is a convex body.

Corollary — The reachable region $C(t)$ is closed when B is either a reflexive space or it can be considered as a conjugate of some other Banach space.

PROOF : See Chaudhuri and Mukherjee (1981).

To solve the minimum time control problem, we shall first consider the following auxiliary problem.

Auxiliary Problem—Let $\xi \in \delta C(t)$ where $\delta C(t)$ denotes the boundary of $C(t)$ for some given time t. Then determine $u \in U$ such that $Tu = \xi$ and $\|u\|$ is minimum. We shall call this the minimum norm problem. Theorems 2, 3 and 4 below are proved in Chaudhuri and Mukherjee (1981).

Theorem 2 — An admissible control which will be optimal in the above sense must satisfy $\|u\| = 1$.

Theorem 3 — Let $\xi \in \delta C(t)$ and $\phi \in D^*$ determine the supporting hyperplane to $C(t)$ at ξ. The $\langle \xi, \phi \rangle = \| T^*\phi \|$, where D^* is conjugate space to D and T^* is the transformation adjoint to T.

Theorem 4 — Let K be a weakly compact set, in a Banach space D and let ϕ be any element $\in D^*$, the conjugate space to D. Then there exists a point $\eta_0 \in \delta K$, such that ϕ defines a supporting hyperplane to K at $\eta_0 \in \delta K$.

Theorem 5 — If $\langle \xi, \phi \rangle = \| T^* \phi \|$ for some $\xi \in C(t)$ and some $\phi \in D^*$, then $\xi \in \delta C(t)$ and ϕ defines a supporting hyperplane to $C(t)$, at ξ, where B is either reflexive or it can be considered as the conjugate of some other Banach space.

PROOF : By the hypothesis made on B, $C(t)$ is weakly compact (see corollary of Theorem 1). Also $C(t)$ is convex (by Theorem 1). Since $\phi \in D^*$ hence by Theorem 4, we can show that there exists a point $\eta \in \delta C(t)$ such that ϕ defines a supporting hyperplane to $C(t)$ at η.

Consequently by Theorem 3, $\langle \eta, \phi \rangle = \| T^* \phi \|$.

But, by hypothesis $\langle \xi, \phi \rangle = \| T^* \phi \|$ for some $\xi \in C(t)$ and $\phi \in D^*$.

$\therefore$ If $\xi \notin \delta C(t)$, then $\langle \xi, \phi \rangle < \langle \eta, \phi \rangle$.

$\therefore$ $\langle \xi, \phi \rangle < \| T^* \phi \|$ which contradicts the hypothesis.

$\therefore$ ξ must $\in \delta C(t)$.

Also since $\langle \eta', \phi \rangle \leqslant \langle \eta, \phi \rangle = \langle \xi, \phi \rangle$, for $\eta' \in C(t)$

$\therefore$ ϕ also defines a supporting hyperplane at ξ.

Corollary — Let $\xi \in \delta C(t)$ where t is the given terminal time and $\phi \in D^*$ define the supporting hyperplane at ξ. Let u_ϕ be the optimal control in the above sense to reach at ξ. The u_ϕ maximises $\langle u, T^*\phi \rangle$.

PROOF : See Chaudhuri and Mukherjee (1981).

The above results hold even with B and T depending on t.

Theorem 6 — The N. A. S. C. for the point $\xi \in C(t)$ to be in $\delta C(t)$ at the time $t = t_f$ is that $\max\limits_{\phi} \dfrac{\langle \xi, \phi \rangle}{\| T_t^* \phi \|} = 1$ where $\phi \in D^*$

(where $T_t : B_t \to D$ is bounded linear onto transformation and B_t is either a reflexive space or it can be considered as the conjugate of some other Banach space).

Sufficiency : Suppose $\max\limits_{\phi} \dfrac{\langle \xi, \phi \rangle}{\| T_t^* \phi \|} = 1$.

Let the maximum be attained for some $\phi = \phi_\xi \in D^*$. Then $\langle \xi, \phi_\xi \rangle = \| T_{t_f}^* \phi_\xi \|$.

Consequently, by Theorem 5, $\xi \in \delta\, C(t_f)$ and ϕ_ξ defines a supporting hyperplane to $C(t_f)$ at ξ.

Necessity : Let $\xi \in \delta C(t_f)$. Then by Theorem 3, $\langle \xi, \phi_\xi \rangle = \| T_{t_f}^* \phi_\xi \|$ where ϕ_ξ is the supporting hyperplane to $C(t_f)$ at ξ.

$$\therefore \quad \frac{\langle \xi, \phi_\xi \rangle}{\| T_{t_f}^* \phi_\xi \|} = 1. \qquad \text{...(A)}$$

Now, we have to show that (A) gives the maximum value of the L.H.S. for all $\phi \in D^*$. Let ψ be any other functional $\in D^*$. If ψ is a supporting hyperplane to $C(t_f)$ at ξ, then (A) holds. So, let us assume, that $\psi \in D^*$ is not a supporting hyperplane to $C(t_f)$ at ξ.

Now by Theorem 1 and its corollary it can be shown that $C(t_f)$ is convex, weakly compact, closed and bounded set. Consequently, by Theorem 4, corresponding to

$\psi \in D^*$ there exist a $\eta_0 \in C(t_f) \cap \delta C(t_f)$ such that ψ is the supporting hyperplane at η_0.

Hence we have $\langle \xi, \psi \rangle \leqslant \langle \eta_0, \psi \rangle = \| T^*_{t_f} \psi \|$

$\therefore \quad \dfrac{\langle \xi, \psi \rangle}{\| T^*_{t_f} \psi \|} \leqslant 1$. This proves the theorem that $\max\limits_{\psi} \dfrac{\langle \xi, \psi \rangle}{\| T^*_{t_f} \psi \|} = 1$.

Theorems 7 and 8 below are proved in Chaudhuri and Mukherjee (1981).

Theorem 7 — Let $\xi \in C(t_f) \cap \delta C(t_f)$ where $C(t_f)$ is the reachable region. Then $\max\limits_{\psi} \dfrac{\langle \xi, \psi \rangle}{\|T_t^* \psi\|}$ is $\leqslant 1$ or $\geqslant 1$, where $\psi \in D^*$, according as $t \geqslant$ or $\leqslant t_f$. (Here B_t is to be considered as in Theorem 5). Moreover the max is attained at a point $\phi \in D^*$, where ϕ is the supporting hyperplane to $\delta C(t)$ at the intersection with the ray through ξ.

Corollary 1 — If $\xi \in C(t_f)$, $\eta = l$, $\xi \in \delta C(t)$ and $\psi \in D^*$ define the supporting hyperplane at η, then $\langle \xi, \psi \rangle > 0$.

PROOF : We have, $\langle \xi, \psi \rangle = \dfrac{1}{l} \langle \eta, \psi \rangle$.

Since $C(t)$ is convex body, $\theta \in \text{Int } C(t)$, where θ is the null element of D.

Again since $\eta \in \delta C(t)$ and ψ defines the supporting hyperplane to $\delta C(t)$ at η,

$\therefore \quad \langle \eta, \psi \rangle > \langle \theta, \psi \rangle = 0$. Hence $\langle \xi, \psi \rangle > 0$.

It may also be noted that $\langle \eta, \psi \rangle = \| T^*_t \psi \|$ by Theorem 3.

Corollary 2 — Let $\xi \in C(t_f) \cap \delta C(t_f)$, where $C(t_f)$is the reachable region.

Then $\max\limits_{\psi} \dfrac{\langle \xi, \psi \rangle}{\| T^*_t \psi \|} < 1$ or > 1 according as $t > t_f$ or $< t_f$ provided that $\delta C(t_1)$ and $\delta C(t_2)$ are disjoint.

Theorem 8 — Let $t_1 < t_2$ and $T_{t_1} : B_{t_1} \to D$ and $T_{t_2} : B_{t_2} \to D$ are bounded linear onto transformations. Then $C(t_1) \subseteq C(t_2)$ and $\delta C(t_1)$ is disjoint with $\delta C(t_2)$ iff $\| T^*_{t_2} \phi \| > \| T^*_{t_1} \phi \|$ for all $\phi \in D^*$.

(Here B_{t_1} and B_{t_2} are to be considered as in Theorem 5).

Theorem 9 — Let $\xi \in \delta C(t_f) \cap C(t_f)$ and $t \geqslant t_f$.

Then $\max\limits_{\phi} \dfrac{\langle \xi, \phi \rangle}{\| T^*_t \phi \|}$ is a non-increasing function of t, $t \geqslant t_f$.

[Here B_t is to be considered as in Theorem 5].

PROOF : Let $t_f < t_1 < t_2$.

Then from Theorem 7

$$\max_{\phi} \frac{\langle \xi, \phi \rangle}{\| T^*_{t_1} \phi \|} = \frac{\langle \xi, \phi_1 \rangle}{\| T^*_{t_1} \phi_1 \|} \qquad \text{...(1)}$$

where $\phi_1 \in D^*$, defines the supporting hyperplane to the point of intersection of the ray through ξ with $\delta C(t_1)$. Denote this point by ξ_1. Then $\xi_1 = l_1 \xi$ for some $l_1 \geqslant 1$. Let $u_{t_1} \in U_{t_1}$ be the optimal control to reach $\xi_1 = T_{t_1} u_{t_1}$ where U_{t_1} is the unit ball in B_{t_1}.

Since T_{t_1} is the restriction of T_{t_2} on U_{t_1}, we have

$$\xi_1 = T_{t_1} u_{t_1} = T_{t_2} u_{t_1}. \qquad \text{...(2)}$$

By Theorem 3, we also have, $\langle \xi_1, \phi_1 \rangle = \|T^*_{t_1} \phi_1 \|$

Thus from (1) we get

$$\max_{\phi} \frac{\langle \xi, \phi \rangle}{\| T^*_{t_1} \phi \|} = \frac{\langle \xi, \phi_1 \rangle}{\| T^*_{t_1} \phi_1 \|} = \frac{\langle \xi, \phi_1 \rangle}{\langle \xi_1, \phi_1 \rangle} = \frac{\langle \xi, \phi_1 \rangle}{\langle T_{t_1}, u_{t_1}, \phi_1 \rangle}$$

$$= \frac{\langle \xi, \phi_1 \rangle}{\langle T_{t_2} u_{t_1}, \phi_1 \rangle} \cdot \qquad \text{...(3)}$$

Again let $\max_{\phi} \dfrac{\langle \xi, \phi. \rangle}{\| T^*_{t_2} \phi \|} = \dfrac{\langle \xi, \phi_2 \rangle}{\| T^*_{t_2} \phi_2 \|}$

where $\phi_2 \in D^*$, defines supporting hyperplane to $\xi_2 = l_2 \xi_1$, for some $l_2 \geqslant l_1$ and $\xi_2 \in \delta C(t_2)$.

Then we obtain similarly as before

$$\max_{\phi} \frac{\langle \xi, \phi \rangle}{\| T^*_{t_2} \phi \|} = \frac{\langle \xi, \phi_2 \rangle}{\| T^*_{t_2} \phi_2 \|} = \frac{\langle \xi, \phi_2 \rangle}{\langle T_{t_2} u_{t_2}, \phi_2 \rangle} \qquad \text{...(4)}$$

where $u_{t_2} \in U_{t_2} \subset B_{t_2}$ is the optimal control to reach at ξ_2.

Now, $\xi_1 \in C(t_1) = T_{t_1} U_{t_1} = T_{t_2} U_{t_1} \subset T_{t_2} U_{t_2} = C(t_2)$.

Since ϕ_2 is the supporting hyperplane to $C(t_2)$ at ξ_2, therefore we have

$$\langle \xi_1, \phi_2 \rangle \leqslant \langle \xi_2, \phi_2 \rangle = \langle T_{t_2} u_{t_2}, \phi_2 \rangle$$

Thus,

$$\langle T_{t_2} u_{t_1}, \phi_2 \rangle \leqslant \langle T_{t_2} u_{t_2}, \phi_2 \rangle \text{, by (2)} \qquad \text{...(5)}$$

But $$\langle T_{t_2} u_{t_1}, \phi_2 \rangle = \langle \xi_1, \phi_2 \rangle = l_1 \langle \xi, \phi_2 \rangle = \frac{1}{l_2} \langle \xi_2, \phi_2 \rangle > 0 \qquad \text{...(6)}$$

since $\theta \in \text{Int}\, C(t_2)$ and $\frac{1}{l_2} > 0$. This also follows from the fact that $\langle \xi_2, \phi_2 \rangle = \| T^*_{t_2} \phi_2 \|$ from Theorem 3. Also,

$$\langle \xi, \phi_2 \rangle = \frac{1}{l_1 l_2} \langle \xi_2, \phi_2 \rangle > 0 \qquad \dots(7)$$

Hence from (5), (6) and (7)

$$\frac{\langle \xi, \phi_2 \rangle}{\langle T_{t_2} u_{t_1}, \phi_2 \rangle} \geqslant \frac{\langle \xi, \phi_2 \rangle}{\langle T_{t_2} u_{t_2}, \phi_2 \rangle}. \qquad \dots(8)$$

Now from (3) and (8)

$$\max_{\phi} \frac{\langle \xi, \phi \rangle}{\| T^*_{t_1} \phi \|} = \frac{\langle \xi, \phi_1 \rangle}{\langle T_{t_2} u_{t_1}, \phi_1 \rangle} \geqslant \frac{\langle \xi, \phi_2 \rangle}{\langle T_{t_2} u_{t_1}, \phi_2 \rangle} \qquad \dots(9)$$

since max is attained at ϕ_1.

Now using (8) and (9) we have

$$\max_{\phi} \frac{\langle \xi, \phi \rangle}{\| T^*_{t_1} \phi \|} \geqslant \frac{\langle \xi, \phi_2 \rangle}{\langle T_{t_2} u_{t_1}, \phi_2 \rangle} \geqslant \frac{\langle \xi, \phi_2 \rangle}{\langle T_{t_2} u_{t_2}, \phi_2 \rangle}$$

$$= \max_{\phi} \frac{\langle \xi, \phi \rangle}{\| T^*_{t_2} \phi \|}.$$

This proves the theorem.

Corollary—$\max_{\phi} \frac{\langle \xi, \phi \rangle}{\| T^*_t \phi \|}$ is a non-increasing function of t, for $t \geqslant 0$. The proof is obvious, since t_f has been choosen arbitrarily.

We shall now consider the question of global controllability of the system. That is we like to investigate the possibility of reaching any point $\eta \in D$ by applying a control $u \in U_t$ where U_t is the unit ball in B_t, such that t is the minimum time taken.

To resolve this question, let us first consider the reachable region $C(t)$ by applying all $u \in U_t \subset B_t$ i.e. $T_t U_t = C(t)$, where T_t is a linear bounded onto transformation from B_t onto D. Now, let $\eta \notin C(t)$ and let $\xi \in \delta C(t)$ be on the ray through η i.e. $\xi = l\,\eta$ where $0 < l < 1$ and t^* be the minimum time to reach ξ. Hence by Theorem 6, we can write

$$\max_{\phi} \frac{\langle \xi, \phi \rangle}{\| T^*_{t^*} \phi \|} = 1 \qquad \dots(B)$$

where $\xi \in \delta C(t^*) \cap \delta C(t)$.

Suppose maximum is attained at $\phi = \phi_1$. Then $\langle \xi, \phi_1 \rangle = \|T^*_{t*} \phi_1\| > 0$ by Corollary 1 of Theorem 7. Now, $\langle \xi, \phi \rangle$ is a continuous function of ϕ, and since $\langle \xi, \phi_1 \rangle > 0$ there exist a neighbourhood of ϕ_1, such that $\langle \xi, \phi \rangle > 0$ for all ϕ in the neighbourhood of ϕ_1. Put $\langle \xi, \phi \rangle = k_\phi > 0$ in this neighbourhood. Thus $\langle \xi, \frac{\phi}{k_\phi} \rangle = 1$.

Put $\psi = \frac{\phi}{k_\phi}$ in (B), Then from (B) we have $\frac{1}{\min_\psi \| T^*_{t*} \psi\|} = 1$ under the constraint $\langle \xi, \psi \rangle = 1$. Then the minimum root of the equation

$$\min_\psi \| T^*_{t*} \psi \| = 1 \qquad \text{...(C)}$$

where $\langle \xi, \psi \rangle = 1$ will give the minimum time to reach at ξ. Now, $\xi \in \delta C(t^*)$. Here t^* is taken as the minimum root of (C). Obviously t^* is the minimum time to reach at ξ. Let $u_{t*} \in U_{t*} \subset B_{t*}$ be the optimal control to reach at $\xi \in \delta C(t^*)$. Thus $\xi = T_{t*} u_{t*}$.

Hence $l\eta = T_{t*} u_{t*}$

So in order to reach η in time t^* we shall have to apply the control $\frac{u_{t^*}}{l} = v_{t*}$ where

$\| v_{t*} \| = 1/l > 1$. Obviously $v_{t*} \notin U_{t*}$. Now let t^{**} be the minimum time to reach η by applying an admissible control, if such a control exists. Then t^{**} will be greater than t^*, as found above. For, if possible, let $t^{**} \leqslant t^*$. Obviously $t^* \neq t^{**}$ as in that case $\eta \in \delta C(t^*)$, which is not true.

So, let $t^{**} < t^*$. Then by Theorem 7 we can write $\max_\phi \frac{\langle \xi, \phi \rangle}{\| T^*_{t**} \phi \|} > 1$ ...(D)

where $\xi \in \delta C(t^*)$.

$$\text{But } \max_\phi \frac{\langle \xi, \phi \rangle}{\| T^*_{t**} \phi \|} = \frac{\langle \xi, \phi_1 \rangle}{\| T^*_{t**} \phi_1 \|} = \frac{\langle l\eta, \phi_1 \rangle}{\| T^*_{t**} \phi_1 \|} = \frac{l \langle \eta, \phi_1 \rangle}{\| T^*_{t**} \phi_1 \|} = l < 1$$

(by Theorem 6), which contradicts (D).

Consequently $t^{**} \nleqslant t^*$ and hence our assertion that $t^{**} > t^*$ is correct. Hence we have the following theorem.

Theorem 10 — Let $T_t U_t = C(t)$ for any given t, and let $\eta \notin C(t)$.

Let $\xi \in \delta C(t)$ be the point on the ray through η and t^* be the minimum time to reach ξ. If there exists an optimal control $u_t \in U_t$ to reach η in minimum time t^{**}, then $t^{**} > t^*$.

Again by applying Theorem 7, we have for $t = t^*$

$$\max_{\phi} \frac{\langle \xi, \phi \rangle}{\| T^*_{t*} \phi \|} = 1 \text{ i.e. } \max_{\phi} \frac{\langle l\eta, \phi \rangle}{\| T^*_{t*} \phi \|} = 1$$

i.e. $\max_{\phi} \dfrac{\langle \eta, \phi \rangle}{\| T^*_{t*} \phi \|} = 1/l > 1$, where $0 < l < 1$.

Evidently $\max_{\phi} \dfrac{\langle \eta, \phi \rangle}{\| T^*_t \phi \|}$ is also a non-increasing function of t.

Now if there exists a time $t = t'$, such that

$$\max_{\phi} \frac{\langle \eta, \phi \rangle}{\| T^*_{t'} \phi \|} < 1 \quad \text{and also if} \quad \max_{\phi} \frac{\langle \eta, \phi \rangle}{\| T^*_{t'} \phi \|}$$

is a continuous function of t, then by the intermediate value property we can assert that there exists a time $t = t^{**}$, such that

$$\max_{\phi} \frac{\langle \eta, \phi \rangle}{\| T^*_{t**} \phi \|} = 1. \qquad \ldots(E)$$

Now $\eta \in C(t^{**})$. For if, $\eta \notin C(t^{**})$, let $\eta' \in \delta C(t^{**})$ be the point on the ray through η so that $\eta' = l\eta$ for some $l < 1$.

Hence from (E), we obtain

$$\max_{\phi} \frac{\langle \eta', \phi \rangle}{\| T^*_{t**} \phi \|} = l < 1,$$

which contradicts $\max_{\phi} \dfrac{\langle \eta', \phi \rangle}{\| T^*_{t**} \phi \|} = 1$ (Theorem 6). Hence, again from (Theorem 5) $\eta \in \delta C(t^{**})$, and the maximum in (E) will be attained at ψ which defines the supporting hyperplane to $C(t^{**})$ at η.

Therefore we have,

$$\langle \eta, \psi \rangle = \| T^*_{t**} \psi \|.$$

Since $\eta \in \delta\, C(t^{**})$ there exist a $u_\eta \in U_{t^{**}}$ such that $\eta = T_{t^{**}} u_\eta$.

Hence $\langle T_{t^{**}} u_\eta, \psi \rangle = \| T^*_{t**} \psi \|$ or $\langle u_\eta, T^*_{t**} \psi \rangle = \| T^*_{t**} \psi \|$.

Hence by Hahn-Banach theorem u_η can be choosen to be $\overline{T^*_{t**} \psi}$ with $\overline{\| T^*_{t**} \psi \|} = 1$.

It can be similarly proved for $\eta \in \text{Int } C(t)$.

Thus we can state the following theorem.

Theorem 11 — The sufficient conditions for the existence of minimum time control for η as in Theorem 10, are that

(a) There exists a time t_1, such that $\max_{\phi} \frac{\langle \eta, \phi \rangle}{\| T_{t_1}^{*} \phi \|} < 1$

and (b) $\max_{\phi} \frac{\langle \eta, \phi \rangle}{\| T_{t}^{*} \phi \|}$ is a continuous function of t.

Theorem 12 — Necessary condition for existence of admissible optimal control is that $\min_{\psi} \| T_t^{*} \psi \| = 1$ under the constraint $\langle \eta, \psi \rangle = 1$ will have atleast one real positive root.

PROOF : See the discussion before Theorem 10.

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