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ON FINSLER SPACES WITH *T*-TENSOR OF SOME SPECIAL FORMS

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The purpose of the present paper is to study the theory of three dimensional Finsler space with *T*-tensor of some special forms.

1. INTRODUCTION

The three dimensional Finsler spaces have been studied by several authors. Moor (1957) introduced an special orthonormal frame in three dimensional Finsler space. Using this frame Matsumoto (1973) has introduced three main scalars H, I, J and h- and v-connection vectors.

Let $C_{ijk}(x, y)$ be the (h) hv-torsion tensor of a n-dimensional Finsler space F_n with the metric function $L(x,y)$, where x is a point and y is an element of support. Then the v-covariant differentiation, being denoted by $|_k$, of C_{hij} defines the T-tensor T_{hijk} in the following form:

$$T_{hijk} = LC_{hij}|_k + C_{hij}l_k + C_{hik}l_j + C_{hkj}l_i + C_{kij}l_h \qquad \ldots(1.1)$$

where

$$l_i = L^{-1}y_i.$$

Ikeda (1979a) studied the theory of n-dimensional Finsler spaces with the T-tensor of the form

$$T_{hijk} = \rho(h_{hi}h_{jk} + h_{hj}h_{ik} + h_{hk}h_{ij})$$

where ρ is certain scalar and h_{ij} is angular metric tensor.

The purpose of the present paper is to study the theory of three dimensional Finsler spaces with T-tensor of the following forms:

(A) $$T_{hijk} = \rho(h_{hi}h_{jk} + h_{hj}h_{ik} + h_{hk}h_{ij})$$

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(B) $$T_{hijk} = h_{hi}P_{jk} + h_{hj}P_{ik} + h_{hk}P_{ij} + h_{ij}P_{hk} + h_{ik}P_{hj} + h_{jk}P_{hi}$$

and

(C) $$T_{hijk} = a_hC_{ijk} + a_iC_{hjk} + a_jC_{hik} + a_kC_{hij} + h_{hi}Q_{jk} + h_{hj}Q_{ik} + h_{hk}Q_{ij} + h_{ij}Q_{hk} + h_{ik}Q_{hj} + h_{jk}Q_{hi}$$

where P_{ij} and Q_{ij} are components of tensor fields and a_h are the components of a covariant vector. In this paper we shall discuss the properties of these three forms of T-tensors in three-dimensional space.

In a two dimensional Finsler space with Berwald's Moor metric (l^i, m^i), the angular metric tensor and (h) hv-torsion tensor are given by (Ikeda 1979b)

$$h_{ij} = m_im_j, \quad LC_{ijk} = Im_im_jm_k \qquad \ldots(1.2)$$

where I is the principal scalar. From (1.1) and (1.2) it follows that the T-tensor of F_2 is given by

$$LT_{hijk} = I_{;2}\, m_hm_im_jm_k \qquad \ldots(1.3)$$

where

$$I_{;2} = L \frac{\partial I}{\partial y^k} m^k .$$

Hence the T-tensor T_{hijk} of F_2 is of the forms (A), (B) and (C) because ρ, P_{ij}, a_h and Q_{ij} are found from (1.2) and (1.3) in the form

$$\rho = \frac{L^{-1} I_{;2}}{3}, \quad P_{ij} = \frac{L^{-1} I_{;2}}{6} m_im_j ,$$

$$a_i = \frac{I_{;2}}{8I} m_i \text{ and } Q_{ij} = \frac{L^{-1} I_{;2}}{12} m_im_j.$$

The T-tensor of a C-reducible Finsler space or Rander's space or Kropina space is of the form (A) (Matsumoto 1974 and Ikeda 1979a). The form (A) of T-tensor is a special case of form (B) with $P_{ij} = \frac{1}{2} \rho h_{ij}$. Thus the T-tensor of a C-reducible Finsler space or Kropina space or Rander's space is of the special form (B).

Let the metric function $L(x,y)$ of F_n be of the form

$$L(x, y) = {}^*L(x, y) + b_i(x)\, y^i \qquad \ldots(1.4)$$

where $^*L(x, y)$ is a metric function of a C-reducible Finsler space and $b_i(x)$ are components of a covariant vector then the T-tensor of F_n is of the form (B) (Singh and Agrawal 1980).

A Finsler space with Kropina metric $\dfrac{g_{ij}(x)\, y^jy^i}{f_i(x)\, y^i}$ is a C-reducible Finsler space (Matsumoto and Hojo 1978), $f_i(x)$ are components of a covariant vector. Thus T-tensor of a Finsler space with metric $(g_{ij}(x)\, y^iy^j/f_i(x)\, y^i) + b_i(x)\, y^i$ is of the form (B).

If T-tensor vanishes identically then F_n is said to be a Finsler space satisfying T-condition. Let metric function $L(x, y)$ of F_n be of the form (1.4) where $^*L(x, y)$ is a metric function of Finsler space satisfying T-condition. If the metric function of F_n is of the form (1.4). Then the T-tensor of F_n is of the form (C) (Singh and Agrawal 1980).

A Finsler space with metric function $(y^1y^2...y^n)^{1/n}$ satisfies T-condition (Matsumoto and Shimada 1978). Thus the T-tensor of a Finsler space with metric $(y^1y^2...y^n)^{1/n} + b_i(x)\, y^i$ is of the form (C).

After giving fundamental formulae of three dimensional Finsler spaces in section 2, we study in section 3 the theory of three dimensional Finsler spaces whose T-tensor is of the form (A) and obtain the value of ρ in terms of main scalars. In section 4 we deal with the form (B) and obtain the values of scalar components of P_{ij} in terms of main scalars and v-connection vectors. In section 5 the form (C) is studied.

2. Fundamental Formulae

Matsumoto (1973) developed the theory of three dimensional Finsler spaces referring to the orthogonal frame $e^{i}_{(\alpha)}$*, $\alpha = 1, 2, 3$, where $e^{i}_{(1)} = L^{-1}y^i$ and $e^{i}_{(2)}$ is the torsion vector $c^i (= c^{i}_{jk}\, g^{jk})$ divided by its length c and $e^{i}_{(3)}$ is given by

$$e^{i}_{(3)} = e^{ijk}\, e_{(1)j}\, e_{(2)k} \qquad ...(2.1)$$

where $e^{ijk} = g^{-1/2}\, \delta^{ijk}_{123}$, g is the determinant g_{ij}.

The tensor C_{ijk} is written as

$$LC_{ijk} = C_{\alpha\beta\gamma} e_{(\alpha)i} e_{(\beta)j} e_{(\gamma)k} \qquad ...(2.2)$$

where the scalar components $C_{\alpha\beta\gamma}$ of LC_{ijk} are such that

$$C_{1\beta\gamma} = 0,\; C_{222} = H,\; C_{233} = I,\; C_{333} = -C_{223} = J. \qquad ...(2.3)$$

The scalars H, I, J are called the main scalars and satisfy the equation

$$H + I = LC. \qquad ...(2.4)$$

The v-covariant derivatives of the vector $e^{i}_{(\alpha)}$ are given by

$$e^{i}_{(\alpha)}\Big|_j L = \delta_{1\alpha}\delta^{i}_{j} - e^{i}_{(1)}\, e_{(\alpha)j} + \delta^{123}_{1\alpha\beta}\; e^{i}_{(\beta)}\, v_j \qquad ...(2.5)$$

where v_j are the components of the v-connection vector. The scalar components $T_{\alpha\beta}$ of a tensor field $T^{i}_{\cdot j}$ are defined by

*Greek letters α, β, γ ......vary from 1 to 3.

$$T_{\alpha\beta} = T^{i}_{\cdot j}\, e_{(\alpha)i}\, e^{j}_{(\beta)}$$

and the v-covariant derivative $T^{i}_{\cdot j}\big|_{k}$ of $T^{i}_{\cdot j}$ are written in the form

$$T^{i}_{\cdot j}\Big|_{k}.\; L = T_{\alpha\beta;\gamma}\, e^{i}_{(\alpha)} e_{(\beta)j}\, e_{(\gamma)k}. \qquad \ldots(2.6)$$

The scalar components $T_{\alpha\beta;\gamma}$ of $T^{i}_{\cdot j}\big|_{k}.\, L$ are

$$T_{\alpha\beta;\gamma} = L \frac{\partial T_{\alpha\beta}}{\partial y^{k}}\, e^{k}_{(\gamma)} + T_{\mu\beta}\Gamma^{(v)}_{(\mu)\alpha\gamma} + T_{\alpha\mu}\Gamma^{(v)}_{(\mu)\beta\gamma} \qquad \ldots(2.7)$$

and the quantities $\Gamma^{(v)}_{(\alpha)\beta\gamma}$ are such that

$$\Gamma^{(v)}_{(\alpha)\beta\gamma} = -\,\Gamma^{(v)}_{(\beta)\alpha\gamma}, \quad \Gamma^{(v)}_{(1)\beta\gamma} = \delta_{\beta\gamma} - \delta_{1\beta}\delta_{1\gamma} \text{ and } \Gamma^{(v)}_{(2)3\gamma} = v_{\gamma} \qquad \ldots(2.8)$$

where v_{γ} are scalar components of v_{j}.

Since $y_{i} = L\, \dfrac{\partial L}{\partial y^{i}}$, from eqns. (2.2) and (2.6),

we have

$$L^{2}.C_{hij}|_{k} + L.C_{hij}\, e_{(1)k} = C_{\alpha\beta\gamma;\delta}\, e_{(\alpha)h} e_{(\beta)i} e_{(\gamma)j} e_{(\delta)k}. \qquad \ldots(2.9)$$

From (1.1) and (2.2), we get

$$L.T_{hijk} = \{C_{\alpha\beta\gamma;\delta} + C_{\beta\gamma\delta}\delta_{1\alpha} + C_{\alpha\gamma\delta}\delta_{1\beta} + C_{\alpha\beta\delta}\delta_{1\gamma}\}$$
$$\times \{e_{(\alpha)h}\, e_{(\beta)i}\, e_{(\gamma)j}\, e_{(\delta)k}\}. \qquad \ldots(2.10)$$

From (2.2), (2.7) and (2.8), we obtain

$$\left.\begin{aligned} C_{1\beta\gamma;\delta} &= -C_{\beta\gamma\delta},\; C_{222;\delta} = H_{;\delta} + 3Jv_{\delta}, \\ C_{223;\delta} &= -J_{;\delta} + (H-2I)\, v_{\delta} \\ C_{233;\delta} &= I_{;\delta} - 3Jv_{\delta},\; C_{333;\delta} = J_{;\delta} + 3Iv_{\delta} \end{aligned}\right\} \qquad \ldots(2.11)$$

where

$$H_{;\delta} = L(\partial H/\partial y^{i})\; e^{i}_{(\delta)}\,.$$

We shall use the following results of Matsumoto (1973).

Lemma 2.1 — If the components of a tensor field T are positively homogeneous of degree r, then the v-scalar derivatives $T_{\ldots;\gamma}$ of the adopted components $T_{\ldots}$ of the T satisfy the equations $T_{\ldots;1} = rT_{\ldots}$,

Lemma 2.2 — The scalar components v_{γ} of the v-connection vector satisfy the following equations

$$v_1 = 0,\ v_2 = C^{-1}C_{;3} \quad \text{...(2.12)}$$

and

$$\text{(a)} \quad (H - 2I)\, v_2 - 3Jv_3 = J_{;2} + H_{;3}$$

$$\text{(b)} \quad 3Jv_2 + (H - 2I)\, v_3 = I_{;2} + J_{;3}$$

$$\text{(c)} \quad 3Iv_2 + 3Jv_3 = -J_{;2} + I_{;3}. \quad \text{...(2.13)}$$

3. Finsler Space With T-Tensor of the Form (A)

Let F_3 be a three dimensional Finsler space whose T-tensor is of the form (A). Since $\delta_{\alpha\beta} - \delta_{1\alpha}\delta_{1\beta}$ are scalar components of the angular metric tensor h_{ij} *i.e.*

$$h_{ij} = (\delta_{\alpha\beta} - \delta_{1\alpha}\delta_{1\beta})\, e_{(\alpha)i}\, e_{(\beta)j} \quad \text{...(3.1)}$$

we have, in view of the form (A) and eqn. (2.10),

$$\begin{aligned} C_{\alpha\beta\gamma;\delta} &+ C_{\beta\gamma\delta}\delta_{1\alpha} + C_{\alpha\gamma\delta}\delta_{1\beta} + C_{\alpha\beta\delta}\delta_{1\gamma} \\ &= \rho L[(\delta_{\alpha\beta} - \delta_{1\alpha}\delta_{1\beta})(\delta_{\gamma\delta} - \delta_{1\gamma}\delta_{1\delta}) \\ &\quad + (\delta_{\alpha\gamma} - \delta_{1\alpha}\delta_{1\gamma})(\delta_{\beta\delta} - \delta_{1\beta}\delta_{1\delta}) \\ &\quad + (\delta_{\alpha\delta} - \delta_{1\alpha}\delta_{1\delta})(\delta_{\beta\gamma} - \delta_{1\beta}\delta_{1\gamma})]. \end{aligned} \quad \text{...(3.2)}$$

In view of (2.11) and (3.2), we get

$$\left.\begin{aligned} &\text{(i)} \ C_{\alpha\beta\gamma;1} = 0 \\ &\text{(ii)} \ H_{;2} + 3Jv_2 = 3\rho L \\ &\text{(iii)} \ H_{;3} + 3Jv_3 = 0 \\ &\text{(iv)} \ -J_{;2} + (H - 2I)\, v_2 = 0 \\ &\text{(v)} \ -J_{;3} + (H - 2I)\, v_3 = \rho L \\ &\text{(vi)} \ I_{;2} - 3Jv_2 = \rho L \\ &\text{(vii)} \ I_{;3} - 3Jv_3 = 0 \\ &\text{(viii)} \ J_{;2} + 3Iv_2 = 0 \\ &\text{(ix)} \ J_{;3} + 3Iv_3 = 3\rho L \end{aligned}\right\} \quad \text{...(3.3)}$$

Since the components $L.C_{ijk}$ are positively homogeneous of degree zero, equation (i) of (3.3) follows directly from the Lemma (2.1). By virtue of eqns. (2.13), equations (iii) and (iv), (v) and (vi), (vii) and (viii) of (3.3) are identical.

Adding equations (iii) and (vii) of (3.3), we get

$$H_{;3} + I_{;3} = 0$$

which in view of (2.4) leads to $C_{;3} = 0$,

which is equivalent to $v_2 = 0$ by virtue of Lemma 2.2.

From equations (ii), (iv) and (vi) of (3.3), we get

$$J_{;2} = 0 \text{ and } \rho = \frac{L^{-1}}{3} H_{;2} = L^{-1} I_{;2}. \qquad \ldots(3.4)$$

Adding equations (v) and (ix) of (3.3) and using eqn. (2.4), we get

$$\rho = \tfrac{1}{4} C \nu_3. \qquad \ldots(3.5)$$

From (3.4), and (2.4), we get

$$\rho = \tfrac{1}{4} C_{;2}. \qquad \ldots(3.6)$$

From (3.5) and (3.6), we get,

$$\nu_3 = C^{-1} C_{;2}. \qquad \ldots(3.7)$$

Theorem 3.1 — If the T-tensor of a 3-dimensional Finsler space is of the form (A) then

$$J_{;2} = 0 \text{ and } \rho = \frac{L^{-1}}{3} H_{;2} = L^{-1} I_{;2} = \tfrac{1}{4} C_{;2} = \tfrac{1}{4} C \nu_3.$$

Theorem 3.2 — The scalar components of a ν-connection vector of a three dimensional Finsler space with the T-tensor of the form (A) is given by

$$\nu_1 = 0,\ \nu_2 = 0 \text{ and } \nu_3 = C^{-1} C_{;2}.$$

4. The Finsler Space With the T-Tensor of the Form (B)

Let F_n be a n-dimensional Finsler space whose T-tensor is of the form (B). Since T_{hijk} is symmetric in all the indices. We have

$$T_{hijk} - T_{ihjk} = 0 = h_{jk}(P_{hi} - P_{ih})$$

from which it follows that P_{ij} is a symmetric tensor.

Since T_{hijk} and h_{ij} are indicatory tensors, we have

$$T_{0ijk} = 0 = h_{ij} P_{0k} + h_{ik} P_{0j} + h_{jk} P_{0i} \qquad \ldots(4.1)$$

where o means contraction with the element of support y^h.

Contracting (4.1) with g^{jk}, we get

$$(n+1)\, P_{0i} - 2L^{-1} P_{00} l_1 = 0.$$

Contracting it with y^i, we get $P_{00} = 0$ for $n > 2$, from which we obtain $P_{0i} = 0$. Hence we have the following lemma.

Lemma 4.1 — Let F_n be a $n \geqslant 2$ -dimensional Finsler space whose T-tensor is of the form (B), then P_{ij} is a symmetric indicatory tensor.

The ν-curvature tensor S_{hijk} is defined as

$$S_{hijk} = C_{hkm}C^m_{ij} - C_{hjm}C^m_{ik}. \qquad \ldots(4.2)$$

The properties of indicaterized tensor of v-covariant derivative $S_{hijk}|_l$ have been studied by Fukui and Yamada (1979). The indicatorized tensor T_{hijkl} of $S_{hijk}|_l$ is

$$T_{hijkl} = S_{hijk}|_l + L^{-1}(2l_l S_{hijk} + l_h S_{lijk} + l_i S_{hljk} + l_j S_{hilk} + l_k S_{hijl}). \qquad \ldots(4.3)$$

The v-covariant differentiation of (4.2) gives

$$S_{hijk}|_l = C_{hkm}|_l\, C^m_{ij} + C_{hkm}\, C^m_{ij}|_l - C_{hjm}|_l\, C^m_{ik} - C_{hjm}C^m_{ik}|_l. \qquad \ldots(4.4)$$

Using the indicatory properties of C_{ijk}, h_{ij}, P_{ij} and the relations (B), (1.1), (4.3) and (4.4), we get

$$LT_{hijkl} = C_{lij}P_{hk} + C_{hkl}P_{ij} - C_{lik}P_{hj} - C_{hjl}P_{ik} + h_{hk}a_{lij} + h_{ij}a_{lhk} - h_{hj}a_{lik} - h_{ik}a_{lhj} + h_{lh}(a_{kij} - a_{jik}) + h_{lk}(a_{hji} - a_{ijh}) + h_{li}(a_{jhk} - a_{khj}) + h_{lj}(a_{ikh} - a_{hki}) \qquad \ldots(4.5)$$

where

$$a_{ijh} = P_{im}C^m_{jh}.$$

Hence we have the following theorem:

Theorem 4.1 — If the indicaterized tensor of $LC_{hij}|_k$ is of the form (B) then the indicaterized tensor of $S_{hijk}|_l$ is of the form (4.5).

Now we shall obtain the scalar components of LP_{ij} in three dimensional Finsler space.

Let $P_{\alpha\beta}$ be the scalar components of LP_{ij} i.e.

$$LP_{ij} = P_{\alpha\beta}e_{(\alpha)i}\, e_{(\beta)j}. \qquad \ldots(4.6)$$

In view of Lemma (4.1), we have

$$P_{\alpha\beta} = P_{\beta\alpha} \text{ and } P_{1\alpha} = 0. \qquad \ldots(4.7)$$

From (2.10), (3.1) and (4.6), we get

$$C_{\alpha\beta\gamma;\delta} + C_{\beta\gamma\delta}\delta_{1\alpha} + C_{\alpha\gamma\delta}\delta_{1\beta} + C_{\alpha\beta\delta}\delta_{1\gamma} = (\delta_{\alpha\beta} - \delta_{1\alpha}\delta_{1\beta})\, P_{\gamma\delta} + (\delta_{\alpha\gamma} - \delta_{1\alpha}\delta_{1\gamma})\, P_{\beta\delta} + (\delta_{\alpha\delta} - \delta_{1\alpha}\delta_{1\delta})\, P_{\beta\gamma} + (\delta_{\beta\gamma} - \delta_{1\beta}\delta_{1\gamma})\, P_{\alpha\delta} + (\delta_{\beta\delta} - \delta_{1\beta}\delta_{1\delta})\, P_{\alpha\gamma} + (\delta_{\gamma\delta} - \delta_{1\gamma}\,\delta_{1\delta})\, P_{\alpha\beta}. \qquad \ldots(4.8)$$

In view of eqns. (2.11), (3.1), (4.7) and (4.8), we get

$$\left.\begin{array}{rl}
\text{(i)} & H_{;2}+3Jv_2=6P_{22} \\
\text{(ii)} & H_{;3}+3Jv_3=3P_{23} \\
\text{(iii)} & -J_{;2}+(H-2I)\,v_2=3P_{23} \\
\text{(iv)} & -J_{;3}+(H-2I)\,v_3=P_{22}+P_{33} \\
\text{(v)} & I_{;2}-3Jv_2=P_{22}+P_{33} \\
\text{(vi)} & I_{;3}-3Jv_3=3P_{23} \\
\text{(vii)} & J_{;2}+3Iv_2=3P_{23} \\
\text{(viii)} & J_{;3}+3Iv_3=6P_{33}.
\end{array}\right\} \qquad \ldots(4.9)$$

By virtue of eqns. (2.13), equations (ii) and (iii), (iv) and (v), (vi) and (vii) of (4.9) are identical.

From equations (iii) and (vii) of (4.9), eqns. (2.12) and (2.4) we get

$$P_{23}=\tfrac{1}{6}LCv_2=\tfrac{1}{6}LC_{;3}. \qquad \ldots(4.10)$$

Again adding equations (iv) and (viii) of (4.9) and using (2.4), we get

$$LCv_3=P_{22}+7P_{33}. \qquad \ldots(4.11)$$

On the other hand from (i) and (v) of (4.9), we get

$$H_{;2}+I_{;2}=7P_{22}+P_{33}$$

which in view of (2.4) reduces to

$$LC_{;2}=7P_{22}+P_{33}. \qquad \ldots(4.12)$$

Equations (4.11) and (4.12) give

$$P_{22}=\frac{L}{48}\,(7C_{;2}-Cv_3)$$

and

$$P_{33}=\frac{L}{48}\,(7Cv_3-C_{;2}).$$

Hence we have the following theorem:

Theorem 4.2—If the T-tensor of a three dimensional Finsler space is of form (B), then the scalar components $P_{\alpha\beta}$ of the tensor LP_{ij} are given by

$$P_{\alpha\beta}=P_{\beta\alpha}, \quad P_{1\alpha}=0, \quad P_{23}=\tfrac{1}{6}LCv_2$$

$$P_{22}=\frac{L}{48}\,(7C_{;2}-Cv_3)$$

and

$$P_{33}=\frac{L}{48}\,(7Cv_3-C_{;2}).$$

Theorem 4.3 — In order that equations (i), (vi), (viii) of (4.9) and eqn. (4.10) are consistent, the rank of the matrix

$$\left\| \begin{matrix} 6 & 0 & 0 & H_{;2} + 3Jv_2 \\ 1 & 0 & 1 & I_{;2} - 3Jv_2 \\ 0 & 0 & 6 & J_{;3} + 3Iv_3 \\ 0 & 6 & 0 & LCv_2 \end{matrix} \right\| \text{ is three.}$$

5. The Finsler Space With the T-Tensor of the Form (C)

Let F_n be a n-dimensional Finsler space whose T-tensor is of the form (C). Since T_{hijk} is symmetric in all the indices, from the form (C) of T_{hijk} it follows that Q_{ij} is a symmetric tensor. Since T_{hijk} is an indicatory tensor, and is of the form (C), we have

$$a_0 C_{ijk} + h_{ij} Q_{0k} + h_{ik} Q_{0j} + h_{jk} Q_{0i} = 0. \qquad \ldots(5.1)$$

Contracting this with y^i, we get $Q_{00} = 0$.

Contracting (5.1) with g^{jk}, we get

$$a_0 C_i + (n+1)\, Q_{0i} = 0. \qquad \ldots(5.2)$$

From (5.1) and (5.2), we get

$$a_0 \left[C_{ijk} - \frac{1}{(n+1)} \left\{ h_{jk} C_i + h_{ik} C_j + h_{ij} C_k \right\} \right] = 0.$$

Thus :

Theorem 5.1 — If F_n is a Finsler space with T-tensor of the form (C) then either $a_0 = 0$ or F_n is a C-reducible Finsler space.

Case I — If $a_0 \neq 0$, then by Theorem 5.1, F_n is a C-reducible Finsler space and hence there exist a scalar $\bar{\rho}$ such that (Matsumoto 1974).

$$T_{hijk} = \bar{\rho}(h_{hi} h_{jk} + h_{hj} h_{ik} + h_{hk} h_{ij}). \qquad \ldots(5.3)$$

Comparing (5.3) with the form (C) and using the C-reducibility property of F_n, we get

$$Q_{ij} = \frac{\bar{\rho}}{2} h_{hk} - \frac{1}{(n+1)} (a_i C_j + a_j C_i). \qquad \ldots(5.4)$$

Thus we have the following theorem:

Theorem 5.2 — If F_n is a Finsler space with T-tensor of the form (C) and $a_0 \neq 0$ then there exists a scalar $\bar{\rho}$ such that Q_{ij} is given by eqn. (5.4).

In a C-reducible three dimensional Finsler space $H = 3I$, $J = 0$, $v_2 = 0$, $I_{;2} = Iv_3$, $I_{;3} = 0$ and the following lemma holds (Matsumoto 1973).

Lemma 5.1 — If a non-Riemannian Berwald 3-space of scalar curvature is c-reducible, then the space is necessarily locally Minkowskian or of constant curvature.

In view of the Theorem 5.1 and above results we have the following theorems:

Theorem 5.3 — If F_n is a 3-dimensional Finsler space with T-tensor of form (C) and $a_0 \neq 0$ then $H = 3I$, $J = 0$, $v_2 = 0$, $I_{;2} = Iv_3$ and $I_{;3} = 0$.

Theorem 5.4 — If the T-tensor of non-Riemannian Berwald 3-space of scalar curvature is of the form (C) and $a_0 \neq 0$ then the space is necessarily locally Minkowskian or of constant curvature.

Case II — If $a_0 = 0$ then from eqn. (5.2) it follows that $Q_{0i} = 0$.

Let a_α and $Q_{\alpha\beta}$ be scalar components of La_h and LQ_{hi} in F_3 then

$$\left.\begin{aligned} &\text{(a)} \quad La_h = a_\alpha e_{(\alpha)h} \\ &\text{(b)} \quad LQ_{hi} = Q_{\alpha\beta} e_{(\alpha)h} e_{(\beta)i} \end{aligned}\right\} \qquad \ldots(5.5)$$

From the form (C) and eqns. (2.2), (2.10), (3.1) and (5.5), we get

$$\begin{aligned} &C_{\alpha\beta\gamma;\delta} + C_{\beta\gamma\delta}\delta_{1\alpha} + C_{\alpha\gamma\delta}\delta_{1\beta} + C_{\alpha\beta\delta}\delta_{1\gamma} \\ &= L^{-1}(a_\alpha C_{\beta\gamma\delta} + a_\beta C_{\alpha\gamma\delta} + a_\gamma C_{\alpha\beta\delta} + a_\delta C_{\alpha\beta\gamma}) \\ &\quad + (\delta_{\alpha\beta} - \delta_{1\alpha}\delta_{1\beta}) Q_{\gamma\delta} + (\delta_{\alpha\gamma} - \delta_{1\alpha}\delta_{1\gamma}) Q_{\beta\delta} \\ &\quad + (\delta_{\alpha\delta} - \delta_{1\alpha}\delta_{1\delta}) Q_{\beta\gamma} + (\delta_{\beta\gamma} - \delta_{1\beta}\delta_{1\gamma}) Q_{\alpha\delta} \\ &\quad + (\delta_{\beta\delta} - \delta_{1\beta}\delta_{1\delta}) Q_{\alpha\gamma} + (\delta_{\gamma\delta} - \delta_{1\gamma}\delta_{1\delta}) Q_{\alpha\beta}. \end{aligned} \qquad \ldots(5.6)$$

By virtue of eqns. (2.11) and (5.6), we get

$$\left.\begin{aligned} &\text{(i)} \quad H_{;2} + 3Jv_2 = 4L^{-1}a_2H + 6Q_{22} \\ &\text{(ii)} \quad H_{;3} + 3Jv_3 = L^{-1}(-3a_2J + a_3H) + 3Q_{23} \\ &\text{(iii)} \quad -J_{;2} + (H - 2I)\, v_2 = L^{-1}(-3a_2J + a_3H) + 3Q_{23} \\ &\text{(iv)} \quad -J_{;3} + (H - 2I)\, v_3 = L^{-1}(2a_2I - 2a_3J) + Q_{22} + Q_{33} \\ &\text{(v)} \quad I_{;2} - 3Jv_2 = L^{-1}(2a_2I - 2a_3J) + Q_{22} + Q_{33} \\ &\text{(vi)} \quad I_{;3} - 3Jv_3 = L^{-1}(a_2J + 3\, a_3I) + 3\, Q_{23} \\ &\text{(vii)} \quad J_{;2} + 3Iv_2 = L^{-1}(a_2J + 3a_3I) + 3Q_{23} \\ &\text{(viii)} \quad J_{;3} + 3Iv_3 = L^{-1}4a_3J + 6Q_{33} \end{aligned}\right\} \qquad \ldots(5.7)$$

Equations (ii) and (iii), (iv) and (v), (vi) and (vii) of (5.7) are identical in view of eqns. (2.13).

To find the scalar components of a_i and Q_{ij} in terms of main scalars, we define the matrices

$$A = \begin{Vmatrix} 4H & 0 & 6 & 0 & 0 \\ -3J & H & 0 & 3 & 0 \\ 2I & -2J & 1 & 0 & 1 \\ J & 3I & 0 & 3 & 0 \\ 0 & 4J & 0 & 0 & 6 \end{Vmatrix},$$

$$X = \begin{Vmatrix} L^{-1}a_2 \\ L^{-1}a_3 \\ Q_{22} \\ Q_{23} \\ Q_{33} \end{Vmatrix}, \quad Y = \begin{Vmatrix} H_{;2} + 3Jv_2 \\ -J_{;2} + (H - 2I)v_2 \\ I_{;2} - 3Jv_2 \\ J_{;2} + 3Iv_2 \\ J_{;3} + 3Iv_3 \end{Vmatrix}.$$

Equations (i), (iii), (v), (vii) and (viii), of (5.7) can be written in the form of matrix equation

$$AX = Y. \qquad \ldots(5.8)$$

Let $A_\lambda(\lambda = 1, \ldots, 5)$ be the matrix obtained from the matrix A by replacing λth column of A by the column matrix Y. Then from (5.8), we get

$$\left.\begin{aligned} a_2 &= L \det A_1/\det A, \quad a_3 = L \det A_2/\det A \\ Q_{22} &= \det A_3/\det A, \; Q_{23} = \det A_4/\det A \\ \text{and} \quad Q_{33} &= \det A_5/\det A. \end{aligned}\right\} \qquad \ldots(5.9)$$

Theorem 5.5 — If the T-tensor of a three dimensional Finsler space is of the form (C) and $a_0 = 0$ then the scalar components a_α and $Q_{\alpha\beta}$ of La_h and LQ_{hi} are given by eqn. (5.9).

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