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# USE OF HERMITE'S METHOD TO OBTAIN GENERATING FUNCTIONS FOR CLASSICAL ORTHOGONAL POLYNOMIALS

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By using a technique originally due to Hermite simple proofs of the generating functions for the Laguerre and Hermite polynomials have been obtained. A unified treatment for obtaining generating functions by Hermite's method for the classical orthogonal polynomials alongwith the Besssel polynomials is also given.

## 1. INTRODUCTION

Motivated by the method of Hermite which he used to prove the orthogonality of Legendre polynomials, Askey (1978) recently obtained the usual generating function for the Jacobi polynomials. The main purpose of this note is to show that Hermite's method applies equally well to the remaining classical orthogonal polynomials, namely the Laguerre polynomials and the Hermite polynomials. The Laguerre polynomials and the Hermite polynomials are respectively defined by the following Rodrigue's formulae:

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^{x}}{n!} D^n\{e^{-x}x^{n+\alpha}\}, \quad D \equiv d/dx \qquad \ldots(1)$$

$$H_n(x) = (-1)^n \exp(x^2)\, D^n \exp(-x^2). \qquad \ldots(2)$$

Using the method of integration by parts it is fairly easy to obtain the orthogonality conditions, namely

$$\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)} L_m^{(\alpha)}(x)\, dx = 0, \quad m \neq n, \quad \alpha > -1 \qquad \ldots(3)$$

$$\int_{-\infty}^\infty \exp(-x^2)\, H_n(x)\, H_m(x)\, dx = 0, \quad m \neq n. \qquad \ldots(4)$$

## 2. GENERATING FUNCTION FOR LAGUERRE POLYNOMIALS

Consider a generating function

$$f(x, t) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n.$$

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Consider also

$$I_k = \int_0^\infty x^k f(x, t)\, x^\alpha e^{-x} dx.$$

Put $x = (1 + t)\, y$ with $| t | < 1$.

The integral $I_k$ is then

$$I_k = \int_0^\infty y^k (1 + t)^k f(x, t)\, (1 + t)^{\alpha+1} y^\alpha e^{-y} e^{-ty} dy.$$

This would clearly be a polynomial of degree $k$ in $t$ if

$$\begin{aligned} f(x, t) &= (1 + t)^{-\alpha-1}\, e^{ty} \\ &= (1 + t)^{-\alpha-1} \exp\, [xt/(1 + t)]. \end{aligned} \qquad \ldots(5)$$

If $$f(x, t) = \sum_{n=0}^\infty Q_n(x) t^n \qquad \ldots(6)$$

then $Q_n(x)$ is a polynomial of degree $n$ in $x$ and as $I_k$ is going to be a polynomial of degree $k$ in $t$, we must have

$$\int_0^\infty x^k Q_n(x)\, x^\alpha e^{-x}\, dx = 0, \quad n = k + 1, \quad k + 2, \ldots.$$

This is equivalent to the orthogonality of $Q_n(x)$ over $(0, \infty)$ with respect to the weight function $x^\alpha e^{-x}$.

Thus $$Q_n(x) = l_n L_n^{(\alpha)}(x)$$

for some constant $l_n$, since there is only one set of polynomials that are orthogonal with respect to a given weight function after they have been normalized.

From (5) and (6), we obtain

$$Q_n(0) = (-1)^n \frac{(1 + \alpha)_n}{n!}.$$

Also from (1)

$$L_n^\alpha (0) = \frac{(1 + \alpha)_n}{n!}$$

and hence

$$Q_n(x) = (-1)^n L_n^{(\alpha)}(x).$$

Thus we get

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n = (1-t)^{-\alpha-1} \exp\left(\frac{-xt}{1-t}\right), \quad |t| < 1. \qquad \ldots(7)$$

The above arguments also contain the proof of the fact that the orthogonality relation (3) for the Laguerre polynomials holds if they are defined by the generating relation (7).

## 3. Generating Function for Hermite Polynomials

As in section 2, let

$$F(x, t) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!}\, t^n$$

and

$$I_k = \int_{-\infty}^{\infty} x^k F(x, t) \exp(-x^2)\, dx.$$

Set

$$x = y + t.$$

Then

$$I_k = \int_{-\infty}^{\infty} (y + t)^k\, F(x, t) \exp(-2yt - t^2) \exp(-y^2)\, dy.$$

Obviously, this is a polynomial of degree $k$ in $t$ if

$$F(x, t) = \exp(2yt + t^2) = \exp(2xt - t^2). \qquad \ldots(8)$$

If we take

$$F(x, t) = \sum_{n=0}^{\infty} q_n(x)\; t^n \qquad \ldots(9)$$

then $q_n(x)$ is a polynomial of degree precisely $n$ in $x$ which must satisfy

$$\int_{-\infty}^{\infty} x^k q_n(x) \exp(-x^2)\, dx = 0, \quad n = k+1,\ k+2,\ \ldots,$$

in order to have $I_k$ a polynomial of degree $k$ in $t$.

The last condition is equivalent to the orthogonality of $q_n(x)$ with respect to the weight function $\exp(-x^2)$ over $(-\infty, \infty)$.

Hence $q_n(x) = h_n H_n(x)$ for some constant $h_n$, as was done in section 2.

We can select $h_n = 1/n!$; because from (8), (9) and (2), we have

$$q_{2n}(0) = \frac{(-1)^n}{n!},\ H_{2n}(0) = \frac{(-1)^n}{n!}(2n)!;$$

and

$$q'_{2n+1}(0) = 2\,\frac{(-1)^n}{n!},\ H'_{2n+1}(0) = 2\,\frac{(-1)^n}{n!}(2n+1)\,!.$$

Finally, we get

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!}\, t^n = \exp\,(2xt - t^2). \qquad \text{...(10)}$$

As was mentioned in section 2 the above arguments also contain the proof of the orthogonality condition (4) for the Hermite polynomials if they are defined by the generating relation (10).

## 4. A Unified Treatment

Thakare (1972) following Fujiwara (1966) stated the extended Jacobi polynomials $F_n(\alpha, \beta; x)$ defined by the following Rodrigues' formula:

$$F_n(\alpha, \beta; x) = [K_n w(x)]^{-1}\, D^n[X(x)^n w(x)], \qquad \text{...(11)}$$

where the normalized weight function $w(x)$, $X(x)$ and $K_n$ are respectively given by

$$w(x) = (x-a)^{\alpha}(b-x)^{\beta}/(b-a)^{\alpha+\beta+1}B(\alpha+1, \beta+1), \quad (\alpha, \beta > -1) \qquad \text{...(12)}$$

$$X(x) = c(x-a)\,(b-x), \ (c > 0), \qquad \text{...(13)}$$

and

$$K_n = (-1)^n\, n! \qquad \text{...(14)}$$

The polynomials $F_n(\alpha, \beta; x)$ are orthogonal with respect to the weight function (12) over the interval $(a, b)$. The parameter $c$ involved in (13) is connected with another parameter $\lambda$ as follows:

$$\lambda = c(b-a). \qquad \text{...(15)}$$

The attempts of Fujiwara and Thakare explicitly bring out the fact that the difference in specific types of classical orthogonal polynomials originates merely from the choice of parameters $\{\alpha, \beta\}$ and $\{a, b\}$ appearing in the weight function (12) and the quadratic $X(x)$ given by (13). More explicitly, one has (see Thakare 1972, 1977, 1980):

(A) *Jacobi polynomials* : With $-a = b = \lambda = 1$, we have

$$F_n(\alpha, \beta; x) = P_n^{(\beta,\alpha)}(x). \qquad \text{...(16)}$$

(B) *Laguerre polynomials*: When $a = 0$, $\beta = b$ and $\lambda = 1$, we have

$$\lim_{b\to\infty} F_n(\alpha, b; x) = (-1)^n L_n^{\alpha}(x). \quad ...(17)$$

(C) *Hermite polynomials*: With $\beta = \alpha$, $-a = b = \sqrt{\alpha}$, and in view of $\lambda \to 2/\sqrt{\alpha}$ (see Fujiwara 1966 and Thakare 1972, 1980), we get

$$\lim_{\alpha\to\infty} F_n(\alpha, \alpha; x) = H_n(x)/n!. \quad ...(18)$$

(D) *Bessel polynomials*: When $-a = b = \lambda = 1$, we have

$$\lim_{\epsilon\to\infty} \frac{\Gamma(n+1)}{\epsilon^n} F_n(r - \epsilon - 1, \epsilon - 1; 1 + (2x\epsilon/s)) = Y_n(x; r, s) \quad ...(19)$$

where $Y_n(x; r, s)$ are the generalized Bessel polynomials introduced by Krall and Frink (1949).

The above relationships are much more handy and easy to work with than the known relationships between the Jacobi polynomials on one hand and the Laguerre and Hermite polynomials on the other hand (see Szegö 1974, p. 103, 107). Why the method of Hermite becomes applicable in obtaining generating functions for each of the classical orthogonal polynomial set is, in fact, a consequence of the above unifying principle.

Consider a generating function

$$G(x, t) = \sum_{n=0}^{\infty} F_n(\alpha, \beta; x) t^n$$

and look at the integral

$$I_k = \int_a^b x^k G(x, t)\,(x - a)^{\alpha}(b - x)^{\beta}\, dx.$$

Put

$$R = [1 + 2tX'(x) + \lambda^2 t^2]^{1/2} = 1 + \lambda t\,[(2(y - a)/(a - b)) + 1] \quad ...(20)$$

where $X(x)$ is given by (13). Then the integral $I_k$ reduces to

$$I_k = \int_a^b [y + \lambda t(y - a)(y - b)/(a - b)]^k\, G(x, t)\,[1 + \lambda t(y - b)/(a - b)]^{\alpha}$$

$$\times [1 + \lambda t(y - a)/(a - b)]^{\beta}\,[1 + \lambda t(1 + 2(y - a)/(a - b))]\,(y - a)^{\alpha}(b - y)^{\beta}\, dy.$$

This integral is a polynomial of degree $k$ in $t$ if

$$G(x, t) = 2^{\alpha+\beta}(1 + \lambda t + R)^{-\alpha}\,(1 - \lambda t + R)^{-\beta}\, R^{-1} \quad ...(21)$$

where $R$ is given by (20). Thus, if

$$G(x, t) = \sum_{n=0}^{\infty} Q_n(x)\, t^n \qquad \text{...(22)}$$

then $Q_n(x)$ is a polynomial of degree $n$ in $x$ and as $I_k$ is to be a polynomial of degree $k$ in $t$, we must have

$$\int_a^b x^k Q_n(x)\, (x - a)^{\alpha}\, (b - x)^{\beta}\, dx = 0, \quad n = k + 1, k + 2, \ldots.$$

Thus, we have to select

$$Q_n(x) = d_n F_n(\alpha, \beta; x)$$

for some constant $d_n$ as there is only one set of polynomials orthogonal over $(a, b)$ with respect to the normalized weight function (12). But with $x = b$, (21) and (22) yield $Q_n(b) = \lambda^n\, (1 + \beta)_n/n!$; and from (11) to (15), we have $F_n(\alpha, \beta; b) = \lambda^n(1 + \beta)_n/n!$ so that we select $d_n = 1$ for all $n$. And thus, we have

$$\sum_{n=0}^{\infty} F_n(\alpha, \beta; x)\, t^n = 2^{\alpha+\beta}(1 + \lambda t + R)^{-\alpha}\, (1 - \lambda t + R)^{-\beta}\, R^{-1} \qquad \text{...(23)}$$

where $R$ is given by (20); see Fujiwara (1966) and Thakare (1977, 1980) for additional proofs of (23).

In view of (16) one readily obtains the usual generating function for the Jacobi polynomials.

Put $a = 0$, $\lambda = 1$ and $\beta = b$ in (23). Using the relationship (17), we obtain on account of (20) and

$$\lim_{b \to \infty} (1 - t/b)^b = e^{-t} \qquad \text{...(24)}$$

the generating function (7) for the Laguerre polynomials.

Put $\beta = \alpha$, $-a = b = \sqrt{\alpha}$, $(\alpha > 0)$ in (23). We have then, as a consequence of (18), (20) and (24) the generating function (10) for the Hermite polynomials.

Lastly, we consider the case of the Bessel polynomials. Put $-a = b = \lambda = 1$, $\alpha = r - \epsilon - 1$, $\beta = \epsilon - 1$ and replace simultaneously $x$ by $1 + (2x\epsilon/s)$, $t$ by $sw/2\epsilon$ in (23) to obtain in the limit $\epsilon \to \infty$

$$G(x, w) = 2^{r-2}(1 - 2xw)^{-1/2}\, [\, 1 + \sqrt{1 - 2xw}]^{2-r}$$
$$\times \lim_{\epsilon \to \infty} [1 + (sw/2\epsilon T)]^{1+\epsilon-r} \cdot \lim_{\epsilon \to \infty} [1 - (sw/2\epsilon T)]^{1-\epsilon}$$

where $$T = 1 + \sqrt{1 - \frac{sw}{\epsilon}\left(1 + \frac{2x\epsilon}{s}\right) + (sw/2\epsilon)^2}.$$

In view of (19) and after some simplification one obtains

$$\sum_{n=0}^{\infty} \left(\frac{s}{2}\right)^n Y_n(x; r, s) \frac{w^n}{n!} = G(x, w)$$

$$= (1 - 2xw)^{-1/2}[\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 - 2xw}]^{2-r} \exp\left[\frac{s}{2x}(1 - \sqrt{1 - 2xw})\right].$$

The above generating function was first given by Burchnall (1951).

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