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STARLIKENESS OF CLOSE-TO-CONVEX FUNCTION

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Let A denote the class of function $f(z) = z + a_2z^2 + \ldots$ regular in the unit disc $E = \{z : |z| < 1\}$. P' will stand for the subclass of A whose elements satisfy $\text{Re } f'(z) > 0$ in E. It is known that P' is a subclass of the class of close-to-convex functions and that P' is not a subclass of the class of starlike functions in E. In this paper we prove that certain subclasses of P' are starlike in E.

Let A denote the class of functions f, which are regular in the open unit disc $E = \{z : |z| < 1\}$ and are normalized so that $f(0) = 0, f'(0) = 1$. By S we designate the subclass of A consisting of univalent functions in E; S^* and C will stand for the subclasses of S whose elements are starlike (w.r.t. the origin) and close-to-convex in E, respectively. Let P' be the subclass of A whose members f satisfy $\text{Re } f'(z) > 0$ in E. It is well known that P' is a subclass of C. Zomorovič (1952) but the question whether P' was a subclass of S^*. Krzyż (1962) gave an example of a function $f \in P'$ such that $f \notin S^*$. The problem of determining the radius of starlikeness of P' is one of the open problems in the theory of univalent functions (see Goodman 1968). Singh and Singh (1977) showed that the radius of starlikeness of P' is not less than 0.853. MacGregor (1964) studied the subclass $: = \{f : |f'(z) - 1| < 1, z \in E\}$ of P' and among other things showed that each member of this class maps the disc $|z| < (\frac{4}{5})^{1/2}$ onto a domain which is starlike w.r.t. the origin. Singh (1977) considered the subclass G_α of P' consisting of functions f which satisfy the condition $|f'(z) - 1| < \alpha, 0 \leqslant \alpha < 1, z \in E$ and showed that $G_\alpha \subset S^*$, for $0 \leqslant \alpha \leqslant (\frac{4}{5})^{1/2}$.

In the present paper we answer Zomorovič's (1952) question in the affirmative for some subclasses of P'.

It is well known that if $f \in P'$ then so does its Libera tranform F, defined by

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt \qquad \ldots(1)$$

where $c > -1$. A natural question arises : If $f \in P'$, is the Libera transform (1) of f starlike in E ? Our Theorem 1 below answers this question in the affirmative for some range of c. In fact we have the following:

Theorem 1 — If $f \in P'$, then the function F, defined by (1) belongs to S^* for all c, $-1 < c \leqslant 0$.

We will need the following result, which we state in the form of a lemma, in proving our results:

Lemma 1 (Miller and Mocanu 1980) — Suppose that the function w is regular in E, with $w(0) = 0$. Then if $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_0, we have

(i) $\dfrac{z_0 w'(z_0)}{w(z_0)} = k,$

(ii) $\text{Re}\,(z_0 w''(z_0)/w'(z_0) + 1) \geqslant k,$

where k is real and $k \geqslant 1$.

Part (i) of the Lemma can also be found in Jack (1971).

PROOF OF THEOREM 1 : From the definition of F, we have,

$$\text{Re}\,[zF''(z) + (1 + c)\,F'(z)] = \underset{> 0,}{\text{Re}\,[(1 + c)\,f'(z)]} \qquad \ldots(2)$$

in E for all c, $c > -1$.

We want to prove that (2) implies that F belongs to S^*, that is,

$$\text{Re}\,\frac{zF'(z)}{F(z)} > 0, \quad (z \in E),$$

for all c, $-1 < c \leqslant 0$. Let us define a function w in E by

$$\frac{zF'(z)}{F(z)} = \frac{1 + w(z)}{1 - w(z)}\,. \qquad \ldots(3)$$

Clearly $w(z)$ is regular in E, $w(0) = 0$ and of course $w(z) \neq \pm 1$, z in E. Differentiating (3) logarithmically and simplifying a little we get

$$zF''(z) + (1 + c)\,F'(z) = cF'(z) + \left[\left(\frac{1 + w(z)}{1 - w(z)}\right)^2 + \frac{2zw'(z)}{(1 - w(z)^2}\right]\left(\frac{F(z)}{z}\right). \qquad \ldots(4)$$

We claim that $|w(z)| < 1$, $z \in E$. Suppose, if possible, there exists a point z_0 in E such that $\max\limits_{|z| \leqslant |z_0|} |w(z)| = |w(z_0)| = 1$. Then by (i) of Lemma 1 we can write $z_0 w'(z_0)/w(z_0) = k$, $k \geqslant 1$. At the point z_0 with $w(z_0) = e^{i\theta}$, $0 \leqslant \theta < 2\pi$, we obtain from (4)

$$\text{Re}\,[z_0 F''(z_0) + (1 + c)\,F'(z_0)]$$
$$= \text{Re}\left\{cF'(z_0) + \left[\left\{\frac{1 + e^{i\theta}}{1 - e^{i\theta}}\right\}^2 + \frac{2ke^{i\theta}}{(1 - e^{i\theta})^2}\right]\left(\frac{F(z_0)}{z_0}\right)\right\}. \qquad \ldots(5)$$

It is readily seen that for all θ, $0 \leqslant \theta < 2\pi$, the expression in the square brackets in the right-hand side of (5) is a negative real number. Also since $F \in P'$, we have $\text{Re}\,(F(z)/z) > 0$ in E. Thus (5) contradicts our hypothesis (2) for all c, $-1 < c \leqslant 0$. This implies that $|\,w(z)\,| < 1$ in E and so $F \in S^*$.

We are unable to prove whether F is starlike in E for $c > 0$.

Letting $c = 0$ in Theorem 1 we have:

Corollary 1 — If $f \in P'$, then the function g, defined by

$$g(z) = \int_0^z \frac{f(t)}{t}\,dt$$

belongs to S^*.

The result of Corollary 1 was obtained by the authors (Singh and Singh 1980).

Theorem 2 — If $f \in A$, $f(z) = z + a_2 z^2 + \ldots$, and satisfies $|\,zf''(z)\,| < 1$ in E, then f is close-to-convex in E, and $|\,a_2\,| \leqslant \frac{1}{2}$. Moreover, if $|\,a_2\,| \leqslant \lambda_0/2$, where $\lambda_0 \doteqdot .743$, is the unique positive root of the transcendental equation.

$$J(\lambda) = \frac{1}{\lambda} - \frac{1-\lambda^2}{\lambda^2}\log\,(1+\lambda) - \frac{2}{\sqrt{5}}$$

then f is starlike univalent in E, that is, $f \in S^*$.

PROOF: Since $zf''(z)$ vanishes at the origin and is bounded by 1 in E, there exists a regular function $\varphi(z) = b_0 + b_1 z + \ldots$, $b_0 = 2a_2$, satisfying $|\,\varphi(z)\,| \leqslant 1$ in E such that

$$zf''(z) = z\varphi(z). \qquad \ldots(6)$$

The assertion that $|\,a_2\,| \leqslant \frac{1}{2}$ now readily follows. Integrating the relation (6) along the linear segment connecting the origin and z we have

$$\begin{aligned} |\,f'(z) - 1\,| &= \Big|\int_0^z f''(z)\,d\,\Big| \\ &= \Big|\int_0^z \varphi(z)\,dz\,\Big| \leqslant \int_0^r |\,\varphi(z)\,|\,dr \leqslant r < 1, \quad (\,|\,z\,| = r) \end{aligned} \qquad \ldots(7)$$

from which it follows that $\text{Re}\,f'(z) > 0$ in E and hence $f \in P' \subset C$. This proves our first assertion. To prove the second assertion we note that since the function φ satisfies the inequality

$$|\,\varphi(z)\,| \leqslant \frac{|\,b_0\,| + |\,z\,|}{1 + |\,b_0\,||\,z\,|}$$

we can write (7) as

$$| f'(z) - 1 | \leqslant \int_0^r \frac{\lambda + p}{1 + \lambda p} \, dp \qquad (| b_0 | = \lambda)$$

$$= \frac{r}{\lambda} - \frac{1 - \lambda^2}{\lambda^2} \log (1 + \lambda r)$$

$$= H(\lambda, r), \text{ say.}$$

One can readily see that $H(\lambda, r)$ is a monotonic increasing function of r and as such its maximum value w.r.t. r is given by

$$H(\lambda, 1) = \frac{1}{\lambda} - \frac{1 - \lambda^2}{\lambda^2} \log (1 + \lambda).$$

The function $H(\lambda, 1)$ is again seen to be a monotonic increasing function of λ. In view of Singh's result f will belong to S^* as long as $H(\lambda, 1)$ is less than or equal to $(\frac{4}{5})^{1/2}$, or equivalently

$$J(\lambda) \equiv \frac{1}{\lambda} - \frac{1 - \lambda^2}{\lambda^2} \log (1 + \lambda) - 2/\sqrt{5} \leqslant 0,$$

which will be true for all λ with $\lambda \leqslant \lambda_0$, where $\lambda_0 \doteqdot .743$, is the unique root of the equation $J(\lambda) = 0$. This completes the proof of Theorem 2.

Theorem 3 — Let f belong to A, $f(z) = z + a_2 z^2 + \ldots$, and satisfy $| zf''(z) | < \alpha$, $\alpha \leqslant (\frac{4}{5})^{1/2} (2 + c)/(1 + c)$, $c > -1$, $z \in E$. Then F, defined by (1), belongs to P' and maps E onto a domain starlike w.r.t. the origin.

PROOF : Since $| zf''(z) | < \alpha$, we have from the definition of F,

$$\left| \frac{z^2 F'''(z) + (2 + c) zF''(z)}{(1 + c)} \right| < \alpha, \qquad (z \in E). \qquad \ldots(8)$$

In view of Singh's result (1977) it suffices to show that the inequality (8) implies the relation.

$$| F'(z) - 1 | < (\tfrac{4}{5})^{1/2} = \beta, \text{ say, } (z \in E). \qquad \ldots(9)$$

Let us define a function w in E by

$$F'(z) = 1 + \beta w(z). \qquad \ldots(10)$$

Evidently $w(0) = 0$, w is regular in E and $w(z) \neq -1/\beta$, $z \in E$.
From (10) we get

$$\frac{z^2 F'''(z) + (2 + c) zF''(z)}{1 + c}$$

$$= \frac{\beta}{1 + c} \cdot zw'(z) \left[\frac{zw''(z)}{w'(z)} + 2 + c \right]. \qquad \ldots(11)$$

We claim that $|w(z)| < 1$ in E. Suppose, if possible, there is a point z_0 in E such that $\max_{|z| \leqslant |z_0|} |w(z)| = |w(z_0)| = 1$. Then by Lemma 1 we can have $z_0 w'(z_0)/w(z_0) = k$, and $\text{Re}\,(z_0 w''(z_0)/w'(z_0) + 1) \geqslant k$, where $k \geqslant 1$. Thus from (11) at the point z_0, we obtain

$$\left| \frac{z_0^2 F'''(z_0) + (2 + c) z_0 F''(z_0)}{1 + c} \right|$$

$$= \left| \frac{\beta}{1 + c} z_0 w'(z_0) \left[\frac{z_0 w''(z_0)}{w'(z_0)} + 2 + c \right] \right|$$

$$\geqslant \frac{\beta k}{1 + c} \text{Re}\,(z_0 w''(z_0)/w'(z_0) + 2 + c)$$

$$\geqslant \frac{\beta}{1 + c} (k + 1 + c)$$

$$\geqslant \beta \frac{2 + c}{1 + c} \geqslant \alpha.$$

This is a contradiction of (8) and so $|w(z)| < 1$ in E. This in turn implies the inequality (9) and hence F belongs so S^*. That $\text{Re}\, F'(z) > 0$ easily follows from (9). This completes the proof of our theorem.

Taking $c = 1$ in Theorem 3, we have :

Corollary 2 — If $f \in A$ and $|zf''(z)| < \alpha$, $\alpha \leqslant 3/\sqrt{5}$, then the function F defined by

$$F(z) = (2/z) \int_0^z f(t)\, dt$$

belongs to S^*.

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