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ON CERTAIN CLASSES OF FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

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Let E denote the unit disc in the complex plane $\mathbb{C}$ and let $p(z)$ be regular in E. We say that $p(z)$ belongs to the class P_k^λ (α) if $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\text{Re}\,\{e^{i\lambda}\, p(z) - \alpha \cos \lambda\}}{1-\alpha} \right| d\theta < k\pi \cos \lambda$$

where $k \geqslant 2$, $0 \leqslant \alpha < 1$, λ real, $|\lambda| < \pi/2$, $z = re^{i\theta}$, $0 \leqslant r < 1$. Let V_k^λ (α, b) denote the class of functions $f(z)$ regular in E with the normalization properties $f(0) = 0$, $f'(0) = 1$ and $1 + b^{-1}zf''(z)/f'(z) \in P_k^\lambda$ (α) where k, α and λ as above and $b \neq 0$ a complex number. Also we define another class U_k^λ (α) of functions for which $zf'(z)/f(z) \in P_k^\lambda$ (α). In this note we generalize both those functions $f(z)$ which are convex of order α with bounded boundary rotation and those functions $f(z)$ for which $zf'(z)$ is λ-spiral-like of order α.

1. INTRODUCTION

Let P_k^λ (α) denote the class of regular functions $p(z)$ in $E = \{z \in \mathbb{C} / \, |z| < 1\}$ with the following properties:

(i) $p(0) = 1$

(ii) $$\int_0^{2\pi} \left| \frac{\text{Re}\,\{e^{i\lambda}p(z) - \alpha \cos \lambda\}}{1-\alpha} \right| d\theta \leqslant k\pi \cos \lambda$$

where $k \geqslant 2$, λ real, $|\lambda| < \pi/2$, $0 \leqslant \alpha < 1$, $z = re^{i\theta}$ and $0 \leqslant r < 1$.

Let V_k^λ (α, b) denote the class of functions $f(z)$ regular in E with the normalization properties $f(0) = 0, f'(0) = 1$ and

$$1 + b^{-1}zf''(z)/f'(z) \in P_k^\lambda\ (\alpha)$$

where k, λ and α are as above and $b \neq 0$ a complex number.

If $\alpha = 0$, $b = 1$ we get the class V_k^{λ} of functions with bounded boundary rotation studied by Moulis (1972). When $\lambda = 0$ and $b = 1$ we obtain the class $V_k(\alpha)$ studied by Padmanabhan and Parvatham (1975) and if $\alpha = 0$ and $\lambda = 0$ the class $V_k(b)$, studied by Nasr (1977). Also, when $b = 1$ the class $V_k^{\lambda}(\alpha)$ was studied by Moulis (1979). For $k = 2$ any function $f(z) \in V_k^{\lambda}(\alpha, b)$ if and only if

$$\text{Re}\,\{e^{i\lambda}(1 + b^{-1}zf''(z)/f'(z))\} > \alpha \cos \lambda.$$

We begin our study by obtaining representation theorem for the class $V_k^{\lambda}(\alpha, b)$. Now we make use of the following lemma.

Lemma 1 (Moulis 1979) — Let $p(z) \in P_k^{\lambda}(\alpha)$. Then

$$e^{i\lambda}p(z) = \frac{\cos \lambda}{2} \int_0^{2\pi} \frac{1 + (1 - 2\alpha)\, ze^{-it}}{1 - ze^{-it}}\, dm(t) + i \sin \lambda$$

where $m(t)$ is a real-valued function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} dm(t) = 2 \text{ and } \int_0^{2\pi} |\, dm(t)\, | \leqslant k.$$

2. Representation Theorem and Some Properties of $V_k^{\lambda}(\alpha, b)$

Theorem 1 — $f(z)$ belongs to $V_k^{\lambda}(\alpha, b)$ if and only if

$$f'(z) = \exp \left\{ -\, b(1 - \alpha)\, e^{-i\lambda} \cos \lambda \int_0^{2\pi} \log\,(1 - ze^{-it})\, dm(t)\right\}$$

where $m(t)$ is as in Lemma 1.

Proof : By Lemma 1, we have

$$e^{i\lambda}\left(1 + \frac{1}{b}\,\frac{zf''(z)}{f'(z)} \right) = \frac{\cos \lambda}{2} \int_0^{2\pi} \frac{1 + (1 - 2\alpha)\, ze^{-it}}{1 - ze^{-it}}\, dm(t) + i \sin \lambda.$$

It follows that

$$\frac{f''(z)}{f'(z)} = be^{-i\lambda} \cos \lambda\, \tfrac{1}{2} \int_0^{2\pi} \left[\frac{1}{z} + \frac{e^{-it}}{1 - ze^{-it}} + \frac{(1 - 2\alpha)\, e^{-it}}{1 - ze^{-it}} \right] dm(t)$$

$$+ \frac{b}{z}\,(e^{-i\lambda}\, i \sin \lambda - 1).$$

Integrating with respect to z we obtain the required expression for $f'(z)$.

Lemma 2 — $f(z)$ is in $V_k^\lambda(\alpha, b)$ if and only if there exists an $f_1(z)$ in $V_k^\lambda(\alpha, 1)$ such that

$$f'(z) = [f_1'(z)]^b.$$

Proof follows at once from Theorem 1.

Lemma 3 — $f(z)$ belongs to $V_k^\lambda(\alpha, b)$ if and only if there exists a function $f_0(z)$ in $V_k^0(\alpha, 1)$ such that

$$f'(z) = [f_0'(z)]\, e^{-i\lambda} b \cos\lambda.$$

PROOF : By Lemma 1 we have

$$1 + \frac{zf_0''(z)}{f_0'(z)} = \tfrac{1}{2}\int_0^{2\pi} \frac{1+(1-2\alpha)\,ze^{-it}}{1-ze^{-it}}\,dm(t) \text{ where } f_0(z) \in V_k^0(\alpha, 1).$$

Thus $$e^{i\lambda}\left[1 + \frac{1}{b}\,\frac{zf''(z)}{f'(z)}\right] = \cos\lambda\left[1 + \frac{zf_0''(z)}{f_0'(z)}\right] + i\sin\lambda.$$

So $$\frac{f''(z)}{f'(z)} = e^{-i\lambda}\cos\lambda\left[\frac{b}{z} + b\frac{f_0''(z)}{f_0'(z)}\right] + b\left[e^{-i\lambda}\, i\sin\lambda - 1\right]\frac{1}{z}.$$

Integrating with respect to z we obtain

$$f'(z) = [f_0'(z)]\, be^{-i\lambda}\cos\lambda.$$

Lemma 4 — $f(z)$ is in $V_k^\lambda(\alpha, b)$ if and only if there exists a function $g_0(z)$ in $V_k^0(0, 1)$ such that

$$f'(z) = [g_0'(z)]^{b(1-\alpha)e^{-i\lambda}\cos\lambda}.$$

PROOF : By Theorem 1, for $f(z) \in V_k^\lambda(\alpha, b)$ if and only if

$$f'(z) = \exp\{-b(1-\alpha)\,e^{-i\lambda}\cos\lambda\int_0^{2\pi}\log(1-ze^{-it})\,dm(t)\}.$$

That is $f'(z) = [\exp\{-\int_0^{2\pi}\log\ (1-ze^{-it})\,dm(t)\}]^{b(1-\alpha)e^{-i\lambda}\cos\lambda}$. Using the representation of elements of $V_k^0(0, 1)$ we recognize that $\exp\{-\int_0^{2\pi}\log\ (1-ze^{-it})\,dm(t)\}$ as the derivative of an element $g_0(z)$ belonging to $V_k^0(0, 1)$. Thus our conclusion follows.

Lemma 5 — $f(z)$ is in V_k^λ (α, b) if and only if there exists a function $g_b(z)$ in V_k^0 $(0, b)$ such that

$$f'(z) = [g_b'(z)]^{(1-\alpha)e^{-i\lambda}\cos\lambda}.$$

Lemma 6 — $f(z)$ is in V_k^λ (α, b) if and only if there exists a function $h(z)$ in V_k^λ $(0, b)$ such that

$$f'(z) = [h'(z)]^{1-\alpha}.$$

Theorem 2 — $f(z)$ belongs to $V_k^\lambda(\alpha, b)$ if and only if there exist two normalized starlike functions $S_1(z)$ and $S_2(z)$ such that

$$f'(z) = \left[\frac{(S_1(z)/z)^{b(k+2)/4}}{(S_2(z)/z)^{b(k-2)/4}} \right]^{(1-\alpha)e^{-i\lambda}\cos\lambda}.$$

Proof follows from Corollary 2 of Nasr (1977) and Lemma 4.

Lemma 7 — If $f(z)$ belongs to V_k^λ (α, b), then $F(z)$ is defined by

$$F'(z) = \frac{f'((z + a)/(1 + \bar{a}z))}{f'(a)(1 + \bar{a}z)^{2b(1-\alpha)e^{-i\lambda}\cos\lambda}}, \ F(0) = 0, \ |z| < 1, \ |a| < 1$$

is also in the class V_k^λ (α, b).

Proof follows by Lemma 5 of Moulis (1979).

Lemma 8 — Suppose $f(z) = z + A_2z^2 + A_3z^3 + \ldots$ belongs to V_k^λ (α, b), then

$$|A_2| \leqslant \frac{k}{2} |b| (1 - \alpha) \cos\lambda$$

and

$$|A_3| \leqslant \frac{1}{3} |b| (1 - \alpha)\cos\lambda \left[1 - |b| (1 - \alpha) \cos\lambda \cdot \frac{k^2}{2}\right].$$

These bounds are sharp, with equality for the function $f(z) \in V_k^\lambda$ (α, b) defined by

$$f'(z) = \left[\frac{(1 - z)^{b(k-2)/2}}{(1 + z)^{b(k+2)/2}} \right]^{(1-\alpha)e^{-i\lambda}\cos\lambda}.$$

PROOF : Let $f(z) = z + A_2z^2 + A_3z^3 + \ldots$ be in V_k^λ (α, b). Then there exists a function $g_0(z) = z + a_2z^2 + a_3z^3 + \ldots$ in V_k^0 (0, 1) such that

$$f'(z) = [g_0'(z)]^{b(1-\alpha)e^{-i\lambda}\cos\lambda}.$$

That is $1 + 2A_2z + 3A_3z^2 + \ldots$

$$= [1 + 2a_2z + 3a_3z^2 + \ldots]^{b(1-\alpha)e^{-i\lambda}\cos\lambda}$$

$$= 1 + b(1-\alpha)\, e^{-i\lambda} 2a_2 \cos\lambda \cdot z + \Big\{ 3b(1-\alpha)\, e^{-i\lambda}\, a_3 \cos\lambda$$

$$+ \frac{b(1-\alpha)\, e^{-i\lambda}\cos\lambda(b(1-\alpha)\, e^{-i\lambda}\cos\lambda - 1)}{2}\, 4a_2^2 \Big\} z^2$$

$$+ \ldots\ldots\ldots.$$

Comparing the coefficients we obtain

$$A_2 = b(1-\alpha)\, e^{-i\lambda}\cos\lambda \,.\, a_2$$

$$A_3 = b(1-\alpha)\, e^{-i\lambda}\cos\lambda\, [a_3 + \tfrac{2}{3}(b(1-\alpha)\, e^{-i\lambda}\cos\lambda - 1)\, a_2^2\,].$$

According to Robertson (1969) we have

$$|\, a_2 \,| \leqslant k/2 \text{ and } |\, a_3 \,| \leqslant (k^2+2)/6.$$

Hence

$$|\, A_2 \,| \leqslant \tfrac{1}{2}k \,|\, b \,|\, (1-\alpha)\cos\lambda$$

$$|\, A_3 \,| = |\, b \,|\, (1-\alpha)\cos\lambda\ (\,|\, a_3 + \tfrac{2}{3}(b(1-\alpha)\, e^{-i\lambda}\cos\lambda - 1)a_2^2 \,|\,)$$

$$\leqslant \tfrac{1}{3}\,|\, b \,|\, (1-\alpha)\cos\lambda\, (1 + |\, b \,|\, (1-\alpha)\cos\lambda\, \tfrac{1}{2}k^2).$$

Theorem 3 — If $f(z)$ belongs to $V_k^\lambda(\alpha, b)$, then it is convex in

$$|z| \leqslant \frac{|b|k(1-\alpha)\cos\lambda - [|b|^2k^2(1-\alpha)^2\cos^2\lambda + 4(2|b|(1-\alpha)\cos\lambda + 1)]^{1/2}}{-2(2\,|\, b \,|\,(1-\alpha)\cos\lambda + 1)}.$$

PROOF : Let $f(z) \in V_k^\lambda(\alpha, b)$. Then $F(z)$ defined by

$$F'(z) = \frac{f'((z+a)/(1+\bar{a}z))}{f'(a)\,(1+\bar{a}z)^{2b(1-\alpha)e^{-i\lambda}\cos\lambda}},\ |\, z \,| < 1,\ |\, a \,| < 1$$

is also in $V_k^\lambda(\alpha, b)$.

$$F''(z) = \frac{1}{f'(a)}\Bigg\{ \frac{(1+\bar{a}z)^{2b(1-\alpha)e^{-i\lambda}\cos\lambda} f''((z+a)/(1+\bar{a}z))(1-|a|^2)/(1+\bar{a}z)^2}{(1+\bar{a}z)^{4b(1-\alpha)\,e^{-i\lambda}\cos\lambda}}$$

$$- \frac{2b(1-\alpha)\, e^{-i\lambda}\cos\lambda\,(1+\bar{a}z)^{2b(1-\alpha)\,e^{-i\lambda}\cos\lambda - 1}\cdot \bar{a}f'((z+a)/(1+\bar{a}z))}{(1+\bar{a}z)^{4b(1-\alpha)e^{-i\lambda}\cos\lambda}}\Bigg\}.$$

$$F''(0) = (1 - |\, a \,|^2)\frac{f''(a)}{f'(a)} - 2b(1-\alpha)\, e^{-i\lambda}\cos\lambda \,.\, \bar{a}.$$

$$\left|\frac{F''(0)}{2}\right| = \frac{1}{2}\left|\frac{f''(a)}{f'(a)}(1-|a|^2) - 2b(1-\alpha)e^{-i\lambda}\cos\lambda\,.\,\bar{a}\right|$$

$$\leqslant \frac{|b|\,k(1-\alpha)\cos\lambda}{2}, \quad |a| < 1.$$

That is $$\left|\frac{f''(a)}{f'(a)} - \frac{2b(1-\alpha)e^{-i\lambda}\cos\lambda\,.\,\bar{a}}{1-|a|^2}\right| \leqslant \frac{|b|\,k(1-\alpha)\cos\lambda}{1-|a|^2}.$$

Replacing a by z in the above inequality and multiplying by $|z|$ we obtain

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2b(1-\alpha)e^{-i\lambda}\cos\lambda\,|z|^2}{1-|z|^2}\right| \leqslant \frac{|b|\,k(1-\alpha)\cos\lambda.\,|z|}{1-|z|^2}$$

for $|z| < 1$

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{1+[2b(1-\alpha)e^{-i\lambda}\cos\lambda - 1]\,|z|^2}{1-|z|^2}\right|$$

$$\leqslant \frac{|b|\,k(1-\alpha)\cos\lambda\,.\,|z|}{1-|z|^2}.$$

For $|z| = r$

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \frac{1+[2b(1-\alpha)e^{-i\lambda}\cos\lambda - 1]\,r^2}{1-r^2}\right\}$$

$$\geqslant -\frac{|b|\,k(1-\alpha)\cos\lambda\,.\,r}{1-r^2}.$$

That is $$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}$$

$$\geqslant \frac{1+\operatorname{Re}[2b(1-\alpha)e^{-i\lambda}\cos\lambda - 1]\,r^2 - |b|\,k(1-\alpha)\cos\lambda\,.\,r}{1-r^2}$$

$$\geqslant \frac{1-|b|\,k(1-\alpha)\cos\lambda\,.\,r - (2\,|b|\,(1-\alpha)\cos\lambda + 1)\,r^2}{1-r^2}.$$

Since $T(r) = 1 - |b|\,k(1-\alpha)\cos\lambda\,.\,r - (2\,|b|\,(1-\alpha)\cos\lambda + 1)\,r^2 = 0$ is a quadratic equation in r, it has got two roots. But clearly one root is negative and another root is positive.

Hence $$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \text{ for}$$

$$|z| = r < \frac{|b|\,k(1-\alpha)\cos\lambda - [\,|b|^2 k^2(1-\alpha)^2\cos^2\lambda + 4(2\,|b|(1-\alpha)\cos\lambda+1)]^{1/2}}{-2(2\,|b|\,(1-\alpha)\cos\lambda + 1)}.$$

Corollary 1 — Let $f(z)$ belong to $V_k^\lambda(\alpha, b)$, then $f(z)$ is univalent in E whenever $0 < |b|\,\cos\lambda\,(2+k)\,(1-\alpha) \leqslant 1$.

PROOF : By the previous theorem we have

$$\left| \frac{f''(z)}{f'(z)} - \frac{2b(1-\alpha)\, e^{-i\lambda} \cos\lambda \; |z|}{1-|z|^2} \right| \leqslant \frac{|b|\, k(1-\alpha)\cos\lambda}{1-|z|^2}, \; |z|<1$$

$$\left| \frac{f''(z)}{f'(z)} \right| \leqslant \frac{|b| \;\cos\lambda (2\,|z| + k)\,(1-\alpha)}{1-|z|^2}$$

$$< \frac{|b| \;\cos\lambda\,(k+2)\,(1-\alpha)}{1-|z|^2} \text{ since } |z|<1.$$

We know that if $\left| \frac{F''(z)}{F'(z)} \right| \leqslant \frac{\beta}{1-|z|^2}$ in $|z|<1$, for some constant β, where β is at least 1, then $F(z)$ is univalent in the unit disc E. Hence $f(z)$ is univalent in E whenever $0 < |b| \cos\lambda\,(1-\alpha)\,(k+2) \leqslant 1$.

Corollary 2 — If $f(z)$ belongs to $V_k^\lambda(0, b)$, then $f(z)$ is univalent in E whenever

$$0 < \cos\lambda < \frac{1}{|b|\,(k+2)}, \; b \neq 0 \text{ complex number.}$$

Theorem 4 — If $f(z)$ belongs to the class $V_k^\lambda(\alpha, b)$, then

$$\frac{zf''(z)}{f'(z)} = \left[\frac{k+2}{4}(p_1(z)-1) - \frac{k-2}{4}(p_2(z)-1) \right] b(1-\alpha)\, e^{-i\lambda} \cos\lambda$$

where $p_i(0) = 1$, Re $p_i(z) > 0$, $i = 1, 2$. That is $p_i(z)$ is in the well known class P consisting of normalized functions which map E onto the right half plane.

PROOF : By Lemma 4, we have for $f(z) \in V_k^\lambda(\alpha, b)$ then there exists $g_0(z)$ in $V_k^0(0, 1)$ such that $f'(z) = \left[g_0'(z) \right]^{b(1-\alpha)\, e^{-i\lambda} \cos\lambda}$.

Taking logarithmic derivative and then multiplying by z we obtain

$$\frac{zf''(z)}{f'(z)} = b(1-\alpha)\, e^{-i\lambda} \cos\lambda z \,\frac{g_0''(z)}{g_0'(z)}.$$

That is $$1 + \frac{1}{b(1-\alpha)} e^{i\lambda} \sec\lambda \frac{zf''(z)}{f'(z)} = 1 + z\frac{g_0''(z)}{g_0'(z)} = \frac{k+2}{4} p_1(z) - \frac{k-2}{4} p_2(z)$$

by Brannan (1968/69).

Hence $$\frac{zf''(z)}{f'(z)} = \left[\frac{k+2}{4}(p_1(z)-1) - \frac{k-2}{4}(p_2(z)-1) \right] b(1-\alpha)\, e^{-i\lambda} \cos\lambda.$$

3. THE CLASS $U_k^\lambda(\alpha)$

A function $f(z)$ regular in E with the normalization properties $f(0) = 0$, $f'(0) = 1$ is said to belong to the class $U_k^\lambda(\alpha)$ if

$$\frac{zf'(z)}{f(z)} \in P_k^\lambda(\alpha).$$

This class generalizes many results obtained previously. Now our immediate result is the following relation between $V_k^\lambda(\alpha, 1)$ and $U_k^\lambda(\alpha)$.

Lemma 9 — $f(z)$ belongs to $V_k^\lambda(\alpha, 1)$ if and only if $zf'(z)$ belongs to $U_k^\lambda(\alpha)$.

Proof follows from the definitions of the two classes.

Theorem 5 — $f(z)$ belongs to $U_k^\lambda(\alpha)$ if and only if

$$f(z) = z \exp\{-(1-\alpha)\, e^{-i\lambda} \cos\lambda \int_0^{2\pi} \log(1 - ze^{-it})\, dm(t)\}$$

where $m(t)$ is a function of bounded variation on $[0, 2\pi]$ satisfying the conditions

$$\int_0^{2\pi} dm(t) = 2 \text{ and } \int_0^{2\pi} |\, dm(t)\, | \leqslant k.$$

Proof is a consequence of Theorem 1.

Theorem 6 — $f(z)$ belongs to $U_k^\lambda(\alpha)$ if and only if there exist two normalized starlike functions $S_1(z)$ and $S_2(z)$ of order α such that

$$f(z) = \left[\frac{(S_1(z))^{(k+2)/4}}{(S_2(z))^{(k-2)/4}}\right]^{e^{-i\lambda}\cos\lambda}.$$

For $k = 2$, a function $f(z) \in U_k^\lambda(\alpha)$ if and only if $\operatorname{Re}\left\{e^{i\lambda} \frac{zf'(z)}{f(z)}\right\} > \alpha\cos\lambda$. If $k = 2$ and $\lambda = 0$, $f(z) \in U_k^\lambda(\alpha)$ if and only if $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha$ and this class coincides with the class of starlike functions of order α. If $\alpha = 0$, $\operatorname{Re}\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\} > 0$ holds for functions $f(z)$ in $U_k^\lambda(\alpha)$ and this class coincides with the class of λ-spiral-like functions.

Lemma 10 — If $f(z) = z + a_2z^2 + \ldots \in U_k^\lambda(\alpha)$, then

$$|\, a_2\, | \leqslant k(1-\alpha)\cos\lambda.$$

Equality holds for the function of the form

$$f(z) = z\left[\frac{(1+z)^{(k-2)/2}}{(1-z)^{(k+2)/2}}\right]^{(1-\alpha)\, e^{-i\lambda}\cos\lambda}.$$

Proof follows by direct calculation.

Theorem 7 — If $f(z) \in U_k^\lambda(\alpha)$, then it is starlike for

$$|z| \leqslant \frac{k(1-\alpha)\cos\lambda - [k^2(1-\alpha)^2\cos^2\lambda + 4(2(1-\alpha)\cos\lambda + 1)]^{1/2}}{-2(2(1-\alpha)\cos\lambda + 1)}.$$

Proof follows from Lemma 9 and Theorem 3.

Theorem 8 — If $f(z)$ belongs to $U_k^\lambda(\alpha)$, then for $|z| = r < 1$

$$r\left[\frac{(1-r)^{(k-2)/2}}{(1+r)^{(k+2)/2}}\right]^{(1-\alpha)\cos\lambda} \leqslant |f(z)| \leqslant r\left[\frac{(1+r)^{(k-2)/2}}{(1-r)^{(k+2)/2}}\right]^{(1-\alpha)\cos\lambda}.$$

PROOF: By Theorem 5, $f(z) \in U_k^\lambda(\alpha)$ we have

$$f(z) = z\exp\{-(1-\alpha)e^{-i\lambda}\cos\lambda\int_0^{2\pi}\log(1 - ze^{-it})\,dm(t)\}.$$

Put $z = re^{i\theta}$, then

$$|f(re^{i\theta})| \leqslant r\exp\{-(1-\alpha)\cos\lambda\int_0^{2\pi}\log|(1-ze^{-it})|\,dm(t)\}. \quad \ldots(3.1)$$

We write $m(t)$ as a difference of two non-decreasing functions. That is $m(t) = m_1(t) - m_2(t)$ on $[0, 2\pi]$ satisfying

$$\int_0^{2\pi} dm_1(t) \leqslant \frac{k}{2} - 1 \text{ and } \int_0^{2\pi} dm_2(t) \leqslant \frac{k}{2} - 1.$$

$$-\int_0^{2\pi}(1-\alpha)\cos\lambda\log|(1-ze^{-it})|\,dm(t)$$

$$= \int_0^{2\pi}(1-\alpha)\cos\lambda\log|(1-ze^{-it})|\,dm_2(t)$$

$$-\int_0^{2\pi}(1-\alpha)\cos\lambda\log|(1-ze^{-it})|\,dm_1(t)$$

$$\leqslant\left(\frac{k}{2} - 1\right)\log(1+r)^{(1-\alpha)\cos\lambda}$$

$$-\left(\frac{k}{2} + 1\right)\log(1-r)^{(1-\alpha)\cos\lambda}$$

$$= \log\left[\frac{(1+r)^{(k-2)/2}}{(1-r)^{(k+2)/2}}\right]^{(1-\alpha)\cos\lambda}.$$

It suffices to prove that

$$-\int_0^{2\pi}(1-\alpha)\cos\lambda\log|(1-ze^{-it})|\,dm(t) \geqslant \log\left[\frac{(1-r)^{(k-2)/2}}{(1+r)^{(k+2)/2}}\right]^{(1-\alpha)\cos\lambda}.$$

If $\int_0^{2\pi} | dm(t) | = k$, then $\int_0^{2\pi} dm_1(t) = \frac{k}{2} + 1$ and $\int_0^{2\pi} dm_2(t) = \frac{k}{2} - 1$.

Thus

$$-\int_0^{2\pi} (1-\alpha)\cos\lambda \log |(1-ze^{-it})| \, dm(t) \geqslant \left(\frac{k}{2} - 1\right) \log (1-r)^{(1-\alpha)\cos\lambda}$$

$$-\left(\frac{k}{2} + 1\right) \log (1+r)^{(1-\alpha)\cos\lambda} = \log \left[\frac{(1-r)^{(k-2)/2}}{(1+r)^{(k+2)/2}}\right]^{(1-\alpha)\cos\lambda}.$$

If $\int_0^{2\pi} | dm(t) | \leqslant k_1 < k$, then

$$-\int_0^{2\pi} (1-\alpha)\cos\lambda \log |(1-ze^{-it})| \, dm(t) \geqslant \log \left[\frac{(1-r)^{(k_1-2)/2}}{(1+r)^{(k_1+2)/2}}\right]^{(1-\alpha)\cos\lambda}$$

$$> \log \left[\frac{(1-r)^{(k-2)/2}}{(1+r)^{(k+2)/2}}\right]^{(1-\alpha)\cos\lambda}.$$

Hence from (3.1) we obtain

$$| f(re^{i\theta}) | \geqslant r \left[\frac{(1-r)^{(k-2)/2}}{(1+r)^{(k+2)/2}}\right]^{(1-\alpha)\cos\lambda}.$$

These bounds are sharp only for the function of the form

$$f(z) = z \left[\frac{(1+z)^{(k-2)/2}}{(1-z)^{(k+2)/2}}\right]^{(1-\alpha)\cos\lambda}.$$

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