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# TORSION OF A NON-HOMOGENEOUS CIRCULAR CYLINDER OF FINITE LENGTH HAVING A RIGID SPHERICAL INCLUSION

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This paper presents a torsion solution, based on Michell-Föppl theory, for a non-homogeneous circular cylinder of finite length having a rigid spherical inclusion. The non-homogeneity arises due to variable rigidity modulus $\mu$. The modulus of rigidity follows the law $\mu = \mu_0 r^{2m}$, $\mu_0$ being a constant, $m$ a positive integer and $r$ denotes the distance of a point in the plane of cross-section from the axis of the cylinder. For different values of $m$ the numerical values of the shear stress in the cylinder has been calculated. To satisfy the boundary condition on the surface of the rigid spherical inclusion the solution in cylindrical polar co-ordinates $(r, \omega, z)$ has been converted to spherical polar co-ordinates $(\rho, \theta, \omega)$. We take the elastic non-homogeneous cylindrical solid of length $2La$ and cross-sectional radius $a$, which has a rigid spherical inclusion of radius $\lambda a$. For a particular case $\lambda = 0.5$, it is found that the effect of rigid spherical inclusion on the shear stress $\tau_{\omega z}$ is 15 per cent less for $m = 0$ (homo case) and for $m = 1$ and 2 (non-homogeneous case) the effect of rigid inclusion geneous on the shear stress is 6 per cent and 2 per cent less respectively compared to the solid cylinder without inclusion.

## INTRODUCTION

Solution of the torsion problems of an infinite homogeneous and/or non-homogeneous circular cylinder having a symmetrically-located spherical cavity and/or a rigid spherical inclusion has been discussed by different authors. Ling (1952), Chattarji (1957), Chatterjee (1964, 1965), Kanoria (1979a, b) and many others have discussed the different problems mentioned above. Now, attention is given to the problem of circular cylinder of finite length instead of infinite circular cylinder. Golovchan (1972) has considered the torsion problem of a circular cylinder of finite length having a spherical cavity. Chattarji and Kanoria (1980) have discussed the torsion problem of a circular cylinder of finite length having a rigid spherical inclusion.

In the present paper the non-homogeneous circular cylinder of finite length having a rigid spherical inclusion has been taken. The non-homogeneity arises due to variable rigidity modulus. The modulus of rigidity follows the law $\mu = \mu_0 r^{2m}$, $\mu_0$ being constant and $m$ being variable, $r$ denotes the distance of a point in the plane of cross-section from the axis of cylinder.

## General Theory and Method of Solution

We assume that an elastic non-homogeneous cylindrical solid of length $2La$ and cross-sectional radius $a$ has a rigid spherical inclusion of radius $\lambda a$ with its centre on the axis and separated from the ends by a distance $La$. Let the tangential forces be applied to the ends which give rise to a torsional stress state in the cylinder and let the lateral surface be free from stress.

We introduce two co-ordinate systems with a common origin at the centre of the inclusion; cylindrical co-ordinates $(r, \omega, z)$ and spherical co-ordinates $(\rho, \theta, \omega)$. For convenience $r$, $z$ and $\rho$ are regarded as dimensionless quantities, referring to the radius of the cylinder and $z$-axis coinciding with the axis of the cylinder.

If $u$, $v$, $w$ be the components of displacement along increasing $r$, $\omega$ and $z$ respectively and we assume $u = w = 0$ and $v$ to be independent of $\omega$, the only two non-vanishing stress components are given by

$$\tau_{r\omega} = \mu r \frac{\partial \phi}{\partial r} \quad \text{and} \quad \tau_{\omega z} = \mu r \frac{\partial \phi}{\partial z} \qquad \ldots(1)$$

where $\mu$ is the modulus of rigidity and $\phi = v/r$ is the angle of rotation of an elemental ring of radius $r$.

In our problem, the modulus being one of varying rigidity, $\mu$ is expressed in the form

$$\mu = \mu_0 r^{2m} \qquad \ldots(2)$$

$\mu_0$ being a constant and $m$ a positive integer.

Equations (1) then assume the form

$$\tau_{r\omega} = \mu_0 r^{2m+1} \frac{\partial \phi}{\partial r} \quad \text{and} \quad \tau_{\omega z} = \mu_0 r^{2m+1} \frac{\partial \phi}{\partial z}. \qquad \ldots(3)$$

Inserting the expressions of eqn. (3) in three stress equations of equilibrium, we see that two of them are identically satisfied and the third reduces to

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{2m+3}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \ldots(4)$$

under certain conditions at all boundaries of the solids.

We represent the function $\phi$ as the sum of two terms $\phi = \phi_0 + \phi_1$ where $\phi_0$ is the solution of the equation (4) for a solid cylinder acted on by an external load and $\phi_1$ takes into account the effect on the rigid spherical inclusion on the stress state. The boundary conditions then take the form

$$\left.\frac{\partial \phi_1}{\partial r}\right|_{r=1} = 0, \quad \left.\frac{\partial \phi_1}{\partial z}\right|_{z=\pm L} = 0, \quad \left.\phi\right|_{\rho=\lambda} = 0 \qquad \ldots(5)$$

Following Kanoria (1979a, b), we construct a certain set of solutions for eqn. (4), $\{\phi^*_{2s}\}$, each of which satisfies the condition in (5). We then construct the function $\phi_1$ from this set by superposition. Considering the symmetry relative to the plane $z = 0$ and the second condition in (5) we write the solution of eqn. (4) in the form of series:

$$\phi^* = \phi_1^{(1)} + \phi_1^{(2)};$$

$$\phi_1^{(1)} = \frac{1}{r^{m+1}} \sum_{n=0}^{\infty} b_n I_{m+1}(x_n r) \sin x_n z, \quad x_n = \frac{(2n+1)\pi}{2L};$$

$$\phi_1^{(2)} = \frac{1}{r^{m+1}} \sum_{k=-\infty}^{\infty} \frac{P_{m+2}^{m+1}(\cos\theta_k)}{\rho_k^{m+3}}. \qquad \ldots(6)$$

Here, $I_{m+1}$ is a modified Bessel function of order $(m+1)$, $b_n$ are arbitrary constants, $P^{m+1}_{m+2}$ is an associated Legendre function; $(\rho_k, \theta_k, \omega)$ is the $k$th spherical co-ordinates system with, origin $0_k$, the co-ordinates of which in $(r, \omega, z)$ system are $(0, 0, 2kL)$ for $k = 0, \pm 1, \pm 2, \ldots$. It is easy to verify that $\phi^*$ satisfies the second condition in (5). Remembering that

$$\frac{1}{\rho_k^{m+3}} P_{m+2}^{m+1}(\cos\theta_k) = \frac{(-1)^{m+1}(2m+3)!\,(z-2kL)\,r^{m+1}}{2^{m+1}(m+1)!\,[(z-2kL)^2+r^2]^{(2m+5)/2}}$$

we obtain from condition $\dfrac{\partial\phi^*}{\partial r} = 0$ when $r = 1$

$$\sum_{n=0}^{\infty} b_n x_n I_{m+2}(x_n) \sin x_n z$$

$$= \frac{(-1)^{m+1}(2m+3)!\,(2m+5)}{2^{m+1}(m+1)!} \sum_{k=-\infty}^{\infty} \frac{(z-2kL)}{[(z-2kL)^2+1]^{(2m+7)/2}}. \qquad \ldots(7)$$

Thus the constant $b_n$ should be chosen so that the quantities equal to the coefficients of the Fourier function on the right side of eqn. (7). After some transformation we have

$$b_n = (-1)^{m+1} \frac{2}{L} x_n^{m+2} \frac{K_{m+2}(x_n)}{I_{m+2}(x_n)}.$$

As a result, we arrive at

$$\phi^* = \frac{1}{r^{m+1}} \frac{2}{L} (-1)^{m+1} \sum_{n=0}^{\infty} x_n^{m+2} \frac{K_{m+2}(x_n)}{I_{m+2}(x_n)} I_{m+1}(x_n r) \sin x_n z$$

$$+ \frac{1}{r^{m+1}} \sum_{k=-\infty}^{\infty} \frac{P_{m+2}^{m+1}(\cos \theta_k)}{\rho_k^{m+3}} \qquad \ldots(8)$$

which is the solution of eqn. (4) satisfying the first two conditions in (5). It is obvious that all even derivatives of $\phi^*$ with respect to $z$ i.e. $\frac{\partial^{2s}\phi^*}{\partial z^{2s}}$, have same properties as well as linear combinations of them. Considering this we represent in the form

$$\phi_1 = \sum_{s=1}^{\infty} \frac{A_{2s}}{(2s-2)\,!} \frac{\partial^{2s-2}\phi^*}{\partial z^{2s-2}} \qquad \ldots(9)$$

where $A_{2s}$'s are parametric coefficients to be determined from the boundary conditions at the surface of the inclusion.

To express the function in terms of spherical polar coordinates the following relations are useful

$$\frac{\partial^{2s-1}}{\partial z^{2s-1}} \left( \frac{1}{\rho_k^{2m+3}} \right) = \frac{(-1)^m\, 2^{m+1}(m+1)\,!\,(2s-1)\,!}{(2m+2)\,!\; r^{m+1}\, \rho_k^{2s+m+1}} P_{2s+m}^{m+1}(\cos \theta_k)$$

(Hobson 1931)

and

$$I_{m+1}(\beta r) \sin \beta z = (-1)^m \sum_{n=1}^{\infty} \frac{(-1)^n (\beta\rho)^{2n+m}}{(2n+2m+1)\,!} P_{2n+m}^{m+1}(\cos \theta)$$

(MacRobert 1947)

and also the additional theorem for spherical harmonics (Hobson 1931)

$$\frac{P_{2s+m}^{m+1}(\cos \theta_k)}{\rho_k^{2s+m+1}} = \sum_{p=m+1}^{\infty} (-1)^{p+m} a_{s,p} \rho^p P_p^{m+1}(\cos \theta)$$

$$\frac{P_{2s+m}^{m+1}(\cos \theta_{-k})}{\rho_{-k}^{2s+m+1}} = \sum_{p=m+1}^{\infty} a_{s,p} \rho^p P_p^{m+1}(\cos \theta)$$

where $$a_{s,p} = \frac{(2s+m+p)\,!}{(p+m+1)\,!\,(2s-1)\,!} \frac{1}{(2kL)^{2s+p+m+1}} \quad (k = 1, 2, \ldots).$$

After several transformations, we have

$$\phi_1 = \frac{1}{r^{m+1}} \sum_{s=1}^{\infty} \left[ \frac{A_{2s}(2s-1)}{\rho^{2s+m+1}} + \rho^{2s+m} \sum_{p=1}^{\infty} \gamma_{p,2s} A_{2p} \right] P_{2s+m}^{m+1} (\cos \theta)$$

$$\gamma_{p,2s} = \frac{2}{(2p-2)!\,(2s+2m+1)!} \left[ (2s+2p+2m)! \sum_{k=1}^{\infty} \frac{1}{(2kL)^{2s+2p+2m+1}} \right.$$

$$\left. + \frac{(-1)^{p+s}}{L} \sum_{n=0}^{\infty} x_n^{2s+2p+2m} \frac{K_{m+2}(x_n)}{I_{m+2}(x_n)} \right]. \qquad \ldots(10)$$

Rewriting the known function $\phi_0$ in these same co-ordinates we obtain from third condition in (5) an infinite system of linear algebraic equations with unknown $A_{2s}$

$$\frac{A_{2s}(2s-1)}{\lambda^{2s+m+1}} + \rho^{2s+m} \sum_{p=1}^{\infty} \gamma_{p,2s} A_{2p} = b_{2s} \quad (s = 1, 2, \ldots). \qquad \ldots(11)$$

Letting the unknown $\frac{A_{2s}(2s-1)}{\lambda^{2s+m+1}} = X_{2s}$, we convert the equation system (11) to the canonical form,

$$X_{2s} + \lambda^{2s+m} \sum_{p=1}^{\infty} \frac{\lambda^{2p+m+1}}{(2p-1)!} \gamma_{p,2s} X_{2p} = b_{2s} \quad (s = 1, 2, \ldots). \qquad \ldots(12)$$

To study the properties of the system (12), we obtain an upper limit for its matrix elements which are defined by second of eqn. (10),

$$\left| \sum_{k=1}^{\infty} \frac{1}{(2kL)^{2s+2p+2m+1}} \right| = \frac{1}{(2L)^{2s+2p+2m+1}} \sum_{k=1}^{\infty} \frac{1}{(k)^{2s+2p+2m+1}}$$

$$< \frac{1}{(2L)^{2s+2p+2m+1}} \, \frac{2^{(2s+2p+2m)}}{2^{(2s+2p+2m)}-1}. \qquad \ldots(13)$$

Evaluation of the second sum in $\gamma_{p,2s}$ is obtained by using asymptotic expressions of $I_{m+2}$ and $K_{m+2}$ (Watson 1952) and the inequality

$$\sum_{n=0}^{\infty} n^{a-1} e^{-bn} < \frac{b^{-a+1}\Gamma(a)}{1-e^{-b}} \quad (a > 0, b > 0)$$

which follows from the improper integral representing the gamma function,

$$\left| \sum_{n=0}^{\infty} \frac{K_{m+2}(x_n)}{I_{m+2}(x_n)} x_n^{2s+2p+2m} \right| < C_1 2^{-(2s+2p+2m)} \cdot (2s+2p+2m)! \qquad \ldots(14)$$

where $C_1$ is a constant.

Taking into account the inequalities (13) and (14) we finally have

$$\left|\gamma_{p,2s}\right| < \frac{2(2s+2p+2m)!}{(2p-2)!\,(2s+2m+1)!}\left[\frac{256}{255}\cdot\frac{1}{(2L)^{2s+2p+2m+1}}\right.$$
$$\left.+\frac{C_1}{2^{2s+2p+2m}}\right]. \qquad \ldots(15)$$

For large values of subscripts $s$ and $p$, the first or second term in the square bracket is predominant depending on the condition $L \leqslant 1$ or $L > 1$. We rewrite the limit (15) in the form

$$\frac{1}{(2p-1)}\left|\gamma_{p,2s}\right| < \frac{C_2(2s+2p+2m)!}{(2p-1)!\,(2s+2m+1)!}\cdot\frac{1}{(2\alpha)^{2s+2p+2m+1}}$$
$$\alpha = \begin{cases} L, & L \leqslant 1 \\ 1, & L > 1 \end{cases}. \qquad \ldots(16)$$

Taking the inequality (16) into account, it is easy to show that the double sum $\sum_{s=1}^{\infty}\sum_{p=1}^{\infty} \mid \gamma_{p,2s} \mid \lambda^{2s+2p+2m+1}$ converges if $\lambda < \alpha$, i.e. under conditions where surface bounding the inclusion is not in contact with the lateral surface of the cylinder or its end planes.

Thus the infinite system of algebraic equation (12) belongs to the class of normal system. It has a unique bounded solution for any bounded right sides $b_{2s}$ if there is no non-zero solution for the corresponding homogeneous system (Kantarovich and Krylov 1958). The sequence $\{b_{2s}\}$ will be bounded practically always. In particular, if

$$\phi_0 = Az,\ b_2 = \frac{(-1)^m(m+1)!\,2^{m+1}}{(2m+3)!}\,\lambda^{m+2}$$

and $b_{2s} = 0$ for $s > 1$, the homogeneous system cannot have a nontrivial solution because of the uniqueness theorem for the boundary value problem under discussion.

The established properties of the system (12) make it possible to use the reduction method for determination of an approximate solution.

Note that the use of the above method also allows one to solve the problem of torsion in a circular cylinder of finite length in the case where the rigid spherical inclusion is located asymmetrically with respect to the ends and also where there are several inclusions with centre in the cylinder axis.

As an illustration, we consider the function $\phi_1$ of the form

$$\phi_1 = A_2\phi^* \qquad \ldots(17)$$

i.e. only the first term of the sum (9). We determine the constant $A_2$ for the system (11)

$$A_2 = \frac{(-1)^m (m+1)!\, 2^{m+1} \lambda^{m+2} A/(2m+3)!}{\lambda^{1/(m+3)} + \lambda^{m+2}\gamma_{1,2}}. \qquad \ldots(18)$$

It is assumed here that $\phi_0 = Az$.

The stress $\tau_{\varphi z}$ reaches maximum at the cross-section $z = 0$, when $r = 1$, it is given by

$$(\tau_{\varphi z})_{\max} = \frac{\mu_0 A}{a^3}\left[1 + \frac{A_2}{A}\left\{\frac{2}{L}(-1)^{m+1}\sum_{n=0}^{\infty} x_n^{m+3}\frac{K_{m+2}(x_n)}{I_{m+2}(x_n)} I_{m+1}(x_n)\right.\right.$$

$$\left.\left. + \frac{(-1)^{m+1}(2m+3)!}{2^{m+1}(m+1)!}\sum_{k=-\infty}^{\infty}\frac{1-(2m+4)\,4k^2L^2}{(4k^2L^2+1)^{(2m+7)/2}}\right\}\right]. \qquad \ldots(19)$$

Thus the effect of inclusion on the maximum stress $\tau_{\varphi z}$ in a non-homogeneous cylinder of finite length is expressed by second term in the braces in eqn. (19). The stress is shown in the third column of the Tables I–III for different values of $m$ and $L = 2$.

TABLE I

$m = 0$

| $\lambda$ | $A_2/A$ | $\tau_{\varphi z}a^3/\mu_0 A$ |
|---|---|---|
| 0.5 | $0.95 \times 10^{-2}$ | 0.8523 |
| 0.51 | $0.104 \times 10^{-1}$ | 0.8383 |
| 0.52 | $0.114 \times 10^{-1}$ | 0.8234 |
| 0.53 | $0.124 \times 10^{-1}$ | 0.8077 |
| 0.54 | $0.135 \times 10^{-1}$ | 0.7908 |
| 0.55 | $0.146 \times 10^{-1}$ | 0.7735 |

TABLE II

$m = 1$

| $\lambda$ | $A_2/A$ | $\tau_{\varphi z}a^3/\mu_0 A$ |
|---|---|---|
| 0.5 | $0.503 \times 10^{-3}$ | 0.9422 |
| 0.51 | $0.57 \times 10^{-3}$ | 0.9341 |
| 0.52 | $0.655 \times 10^{-3}$ | 0.9248 |
| 0.53 | $0.74 \times 10^{-3}$ | 0.9144 |
| 0.54 | $0.84 \times 10^{-3}$ | 0.9029 |
| 0.55 | $0.93 \times 10^{-3}$ | 0.8901 |

TABLE III

$m = 2$

| $\lambda$ | $A_2/A$ | $\tau_{\varphi z}a^3/\mu_0 A$ |
|---|---|---|
| 0.5 | $0.183 \times 10^{-4}$ | 0.9816 |
| 0.51 | $0.218 \times 10^{-4}$ | 0.9778 |
| 0.52 | $0.259 \times 10^{-4}$ | 0.9736 |
| 0.53 | $0.308 \times 10^{-4}$ | 0.9686 |
| 0.54 | $0.362 \times 10^{-4}$ | 0.9631 |
| 0.55 | $0.462 \times 10^{-4}$ | 0.9566 |

From the computational viewpoint the solution proposed above is more convenient than that presented in Chattarji (1957). Note that the improper integrals for the calculation of which the numerical method must be used appear in the solution in Ling (1952). The solution given here contains series which converges rather rapidly.

The result of Chattarji and Kanoria (1980) follows immediately from the result of the present paper by taking $m = 0$.

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