Indian J. pure appl. Math., **13**(2) : 229–236, February 1982

SUPER-CONTINUOUS MAPPINGS

B. M. MUNSHI AND D. S. BASSAN

Department of Mathematics, University School of Sciences,
Gujarat University, Ahmedabad 380009

(*Received* 24 *March* 1981)

A new class of mappings, called 'Super-continuous mappings', has been defined and studied. These maps have been used to discuss some properties of nearly compact spaces (Singal and Mathur 1969) almost regular spaces (Singal and Arya 1969) and almost completely regular spaces (Singal and Arya 1970). Also δ-quotient topology is introduced with the help of such maps.

1. INTRODUCTION

In the literature there are many strong continuities introduced by various authors. The object of the present paper is to introduce a new class of mappings called "Super-continuous mappings". This class is contained in the class of continuous mappings. Super-continuous mappings turn out to be the natural tool for studying nearly-compact spaces (Singal and Mathur 1969), almost regular spaces (Singal and Arya 1969) and almost completely regular spaces (Singal and Arya 1970). Various properties of such mappings have been discussed in section 2. Section 3 is concerned with δ-quotient topology and δ-quotient spaces. The last section is concerned with results on nearly compact spaces, almost regular spaces and almost completely regular spaces.

2. SUPER-CONTINUOUS MAPPINGS

Definition 2.1 — A mapping $f: X \to Y$ is said to be super-continuous at a point $x \in X$ if for every neighbourhood M of $f(x)$ there is a neighbourhood N of x such that $f(\bar{N})^{\circ} \subseteq M$.

Remark 2.1 : It is clear that if $f: X \to Y$ is super-continuous at a point $x \in X$, then it is continuous at x. But the converse is not true.

Example 2.1 — Let $X = \{a, b, c, d\}$ and

$$\mathcal{T}_1 = \{X, \varphi, \{a, b\}, \{a, b, d\}\}.$$

Let $$Y = \{1, 2, 3, 4\} \text{ and}$$

$$\mathcal{T}_2 = \{Y, \varphi, \{1, 2, 3\}, \{1, 3\}\}.$$

Let $f: X \to Y$ be defined as follows

$$f(a) = 1 = f(b), f(c) = 2, f(d) = 3.$$

Then f is continuous at each point of X but f is not super-continuous at $x = a$.

Definition 2.2 — A mapping $f: X \rightarrow Y$ is said to be super-continuous [denoted henceforth as SC] if it is super-continuous at each point of X.

Definition 2.3 (Velicko 1966) — A set G is said to be δ-open if for each $x \in G$, there exists a regularly open set H such that $x \in H \subseteq G$, or equivalently, if G is expressible as an arbitrary union of regularly open sets.

A set is δ-closed, iff its complement is δ-open.

Theorem 2.1 — For a mapping $f: X \rightarrow Y$, the following are equivalent:

(a) f is supercontinuous.

(b) Inverse image of every open subset of Y is a δ-open subset of X.

(c) Inverse image of every closed subset of Y is a δ-closed subset of X.

(d) For each point x of X and for each open neighbourhood M of $f(x)$, there is a δ-open neighbourhood N of x such that $f(N) \subset M$.

PROOF : (a) $\Rightarrow$ (b).

Let U be any open subset of Y and let $x \in f^{-1}(U)$. Then $f(x) \in U$. Therefore, there exists an open set V in X such that $x \in V$ and $f(\bar{V}^{\circ}) \subset U$. Thus $x \in \bar{V}^{\circ} \subset f^{-1}(U)$. Therefore, $f^{-1}(U)$ is expressible as an arbitrary union of regularly open sets. Hence $f^{-1}(U)$ is δ-open.

(b) $\Rightarrow$ (c) obvious.

(c) $\Rightarrow$ (d). Since M is open, therefore $Y \sim M$ is closed and consequently $f^{-1}(Y \sim M)$ is δ-closed. Therefore $f^{-1}(M)$ is δ-open. Also, $x \in f^{-1}(M) = N$ (say). Then N is a δ-open neighbourhood of x such that $f(N) \subset M$.

(d) $\Rightarrow$ (a) Easy.

Corollary 2.1 — Let X and Y be topological spaces. A mapping $f: X \rightarrow Y$ is SC if the inverse image of every open subset of Y is a clopen subset of X.

Definition 2.4 (Stone 1977) — A space is said to be semi-regular if for each point x of the space and each open set U containing x there is a open set V such that

$$x \in V \subset \bar{V}^{\circ} \subset U.$$

Theorem 2.2 — Let $f: X \rightarrow Y$ be a continuous mapping of a semi-regular space X into Y. Then f is SC.

PROOF : Let $x \in X$ and let G be an open set containing $f(x)$. Then $f^{-1}(G)$ is open in X since f is continuous. Therefore there is an open subset M of x such that $x \in M \subset \bar{M}^{\circ} \subset f^{-1}(G)$, since X is semi-regular. Hence f is SC.

Remark 2.2 : Every open set in a T_3-space can be written as the union of regular open sets.

Corollary 2.2 — Let X be a T_3-topological space and let $f : X \to Y$ be continuous; then f is SC.

PROOF : Every regular (or T_3) space is semi-regular.

Theorem 2.3 — Let X and Y be topological spaces. Then a mapping $f : X \to Y$ is SC iff the inverse image under f of every member of a base (sub base) for Y is δ-open in X.

PROOF : Obvious.

Definition 2.5 (Velicko 1966) — A point x is said to be a δ-adherent point of a set P in a space X if the interior of every closed neighbourhood of the point x intersects P or equivalently, every regularly open set containing x has non-empty intersection with P.

Definition 2.6 (Velicko 1966) — The set $(P)_\delta$ of all δ-adherent points of a set P is called the δ-closure of the set P.

Theorem 2.4 — A mapping f from a space X into another space Y is SC iff,

$$f(A)_\delta \subset \overline{f(A)} \text{ for every } A \subset X.$$

PROOF : Let f be SC. Since $\overline{f(A)}$ is closed in Y, $f^{-1}(\overline{f(A)})$ is δ-closed in X, since f is SC.

Now $f(A) \subset \overline{f(A)}$ implies $A \subset f^{-1}(\overline{f(A)})$.

Therefore $(A)_\delta \subset [f^{-1}(\overline{f(A)})]_\delta = f^{-1}(\overline{f(A)})$.

Therefore $f(A)_\delta \subset ff^{-1}(\overline{f(A)}) \subset \overline{f(A)}$.

Conversely, let $f(A)_\delta \subset \overline{f(A)}$ for every $A \subset X$.

Let F be any closed set in Y so that $\overline{F} = F$. Now $f^{-1}(F)$ is a subset of X implies that $f[f^{-1}(F)]_\delta \subset \overline{f[f^{-1}(F)]} \subset \overline{F} = F$ implies that $[f^{-1}(F)]_\delta \subset [f^{-1}(F)]$. Therefore $[f^{-1}(F)]_\delta = f^{-1}(F)$. Hence f is SC.

Theorem 2.5 — A mapping f from a space X into another space Y is SC iff,

$$[f^{-1}(B)]_\delta \subset f^{-1}(\overline{B}) \text{ for every } B \subset Y.$$

PROOF : Let f be SC. Since $\overline{B}$ is closed in Y, $f^{-1}(\overline{B})$ is δ-closed in X. Therefore $f^{-1}(\overline{B}) = [f^{-1}(\overline{B})]_\delta$.

Now $B \subset \overline{B} \subset [B]_\delta$ implies $[f^{-1}(B)]_\delta \subset [f^{-1}(\overline{B})]_\delta$ implies $[f^{-1}(B)]_\delta \subset f^{-1}(\overline{B})$.

Conversely, let the condition hold and let F be any closed set in Y. Therefore $\bar{F} = F$. Now $[f^{-1}(F)]_\delta \subset f^{-1}(\bar{F}) = f^{-1}(F)$.

But $f^{-1}(F) \subset \overline{f^{-1}(F)} \subset [f^{-1}(F)]_\delta$

Hence $f^{-1}(F) = [f^{-1}(F)]_\delta$

Hence f is SC.

Definition 2.6 (Singal and Singal 1968) — A point x is called a δ-adherent point of a filter-base $\mathcal{F}$ iff $x \in \cap \{[F]_\delta : F \in \mathcal{F}\}$.

Definition 2.7 (Singal and Singal 1968) — A filter-base is said to δ-converge to a point x (written as $\mathcal{F} \xrightarrow{\delta} x$) if every regularly open set containing x contains an $F \in \mathcal{F}$.

Theorem 2.6 — Let $f : X \to Y$ be a mapping. Then f is SC at $x \in X$ iff the filter-base $f(\mathcal{U}(x)) \to f(x)$, where $\mathcal{U}(x)$ denotes the δ-nbhd filter at x.

PROOF : Obvious.

Theorem 2.7 — A mapping $f: X \to Y$ is SC on X iff $f(\mathcal{U}) \to f(x)$ for each $x \in X$ and each filter-base $\mathcal{U}$ that δ-converges to x.

PROOF : Assume that f is SC on X and let $\mathcal{U} \xrightarrow{\delta} x$.

Let W be a nbhd of $f(x)$. Then $x \in f^{-1}(W)$ and $f^{-1}(W)$ is δ-open since f is SC. Therefore $x \in H$ such that $f(H) \subset W$ where H is regular open and $H \subset f^{-1}(W)$.

$\therefore \exists$ a $u \in \mathcal{U}$ such that $u \in H$.

Therefore $f(u) \subset f(H) \subset W$.

Therefore $f(\mathcal{U}) \to f(x)$.

Conversely. Let W be any open subset of Y containing $f(x)$. Let B be any subset of X. We have to prove that $f([B]_\delta) \subset \overline{f(B)}$.

Let $b \in [B]_\delta$. Let $\mathcal{U}$ be a filter-base on B with $\mathcal{U}$-δ-converging to b so that $f(\mathcal{U}) \to f(b)$.

Since $f(\mathcal{U})$ is a filter-base on $f(B)$,

therefore $f(b) \in [f(B)]_\delta \subset \overline{f(B)}$ (Singal and Arya 1969). Therefore f is super-continuous.

Theorem 2.8 (Restricting the range) — If $f : X \to Y$ is SC and $f(X)$ is taken with the subspace topology, then $f : X \to f(X)$ is SC.

PROOF : $f : X \to Y$ is SC implies $f^{-1}(U)$ is δ-open, where U is some open subset of Y.

Now $f^{-1}[U \cap f(X)] = f^{-1}(U) \cap f^{-1}[f(X)] = f^{-1}(U) \cap X = f^{-1}(U)$ is δ-open. Therefore $f : X \to f(X)$ is SC.

Theorem 2.9 (Expanding the range) — Let $f : X \to Y$ be SC. If Z is a space having Y as a subspace then the function $h : X \to Z$ obtained by expanding the range of f is SC.

PROOF : We have to show that $h : X \to Z$ is SC. As Z has Y as a subspace, h is the composite of the map $f : X \to Y$ which is SC and the inclusion map $g : Y \to Z$ which is continuous. Thus h is SC.

Theorem 2.10 (Maps into products) — Let $f : A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$. Then f is SC iff the functions $f_1 : A \to X$ and $f_2 : A \to Y$ are SC.

PROOF : Obvious.

Definition 2.8 (Singal and Singal 1968) — A mapping $f : X \to Y$ is said to be almost open if the image of every regularly open subset of X is an open subset of Y.

A mapping $f : X \to Y$ is said to be almost closed if the image of every regularly closed subset of X is a closed subset of Y.

Definition 2.9 (Singal and Singal 1968) — A mapping $f : X \to Y$ is said to be almost continuous at a point $x \in X$ if for every neighbourhood M of $f(x)$ there is a neighbourhood N of x such that $f(N) \subset \bar{M}^{\circ}$.

Theorem 2.11 — If f is an almost open, SC mapping of X onto Y and if g is a mapping of Y into Z then $g \circ f$ is SC iff g if continuous.

PROOF : The sufficiency is clear.

Necessity : Let $g \circ f$ be SC. Let G be an open subset of Z. Therefore $(g \circ f)^{-1}(G)$ is a δ-open subset of X since $g \circ f$ is SC i.e.

$$f^{-1}[g^{-1}(G)] \text{ is a } \delta\text{-open subset in } X.$$

Also as f is almost open and onto, $f[f^{-1}(g^{-1}(G))] = g^{-1}(G)$ is open. Hence g is continuous.

Theorem 2.12 — Let X, Y and Z be topological spaces and the mapping $f : X \to Y$ be almost continuous and $g : Y \to Z$ be SC. Then the composition mapping $g \circ f : X \to Z$ is continuous.

PROOF : Obvious.

But if $f : X \to Y$ is almost continuous and $g \circ f : X \to Z$ is continuous, then $g \circ f : Y \to Z$ need not be SC.

Example 2.2 — Let $(R, \mathcal{T}_1)$ be the topological space, where $\mathcal{T}_1$ is the topology consisting of φ, R and complements of countable subsets of R. Let $X = \{a, b\}$ and

$\mathcal{T}_2 = \{X, \varphi, \{a\}\}$. Let $f : R \to X$ be defined as follows:

$$f(x) = \left.\begin{matrix} a \\ b \end{matrix}\right\} \begin{matrix} \text{if } x \text{ is irrational} \\ \text{if } x \text{ is rational.} \end{matrix}$$

Let $Y = \{1, 2\}$ and $\mathcal{T}_3 = \{Y, \varphi, \{2\}\}$.

Let $g : X \to Y$ be defined as follows:

$g(a) = 2$, $g(b) = 1$. Then $f : R \to X$ is almost continuous and $g \circ f : R \to Y$ is continuous but $g : X \to Y$ is not SC.

3. δ-Quotient Topology and δ-Quotient Spaces

Definition 3.1 — Let f be a mapping of a topological space E onto a set F. The topology on F for which a subset $A \subset F$ is open iff $f^{-1}(A)$ is δ-open in E is called the δ-quotient topology. It is easy to check that the sets A define a topology.

Remark 3.1 — It is clear that the δ-quotient topology depends upon f. Moreover, a subset $B \subset F$ is closed in the δ-quotient topology iff $f^{-1}(B)$ is δ-closed in E.

The following theorem characterizes the δ-quotient topology.

Theorem 3.1 — Let f be a mapping of a topological space (X, U) onto a topological space (Y, V), where V is the δ-quotient topology on Y. Then f is SC and almost open. Moreover V is the finest topology on Y which makes $f : X \to Y$ SC.

Proof : The SC of f follows from the definition of V. Again for each δ-open set $D \subset X$, $D \subset f^{-1} f(D)$ shows that f is almost open by the definition of the δ-quotient topology.

Lastly let W be a topology on Y such that $f : (X, U) \to (Y, W)$ is SC. Let G be a W-open set in Y. By SC of f, $f^{-1}(G)$ is δ-open in X. Now by the definition of the δ-quotient topology on X, G is V-open and hence $W \subset V$.

Corollary 3.1 — If $f : X \to Y$ is a SC, almost open and onto mapping then the topology V on Y is the δ-quotient topology.

Theorem 3.2 — Let (Y, V) be a topological space with the δ-quotient topology defined by a topological space (X, U) and the mapping $f : X \to Y$. Then the mapping $g : (Y, V) \to (Z, W)$, where Z is any topological space is continuous iff the composition $g \circ f : (X, U) \to (Z, W)$ is SC.

Proof : Assume g is continuous.

Since $f : X \to Y$ is always SC

$(g \circ f)^{-1}(G) = f^{-1} [g^{-1}(G)]$ is δ-open. Hence $g \circ f$ is SC.

Conversely. If $g \circ f$ is SC, then for any open set G in Z, $f^{-1}[g^{-1}(G)]$ is δ-open in X. Hence $g^{-1}(G)$ is V-open in Y because of the definition of the δ-quotient topology. Hence g is continuous.

4. Other Results

Definition 4.1 (Singal and Mathur 1969) — A space X is said to be nearly compact if every open cover of X has a finite subfamily, the interiors of the closures of whose members cover X.

Remark 4.1 (Singal and Mathur 1974) : For a space X, X is nearly compact if every δ-open cover has a finite subcover.

Remark 4.2 (Singal and Mathur 1974) : Every regularly closed subset of a nearly compact space is nearly compact.

Theorem 4.1 — Let $f : X \to Y$ be a SC mapping of a nearly compact space X into a topological space Y, then $f(X)$ is compact.

Proof : Let $E = \{0_\alpha : \alpha \in \Lambda\}$ be any open cover of $f(X)$.

Then $f^{-1}(0_\alpha)$ is δ-open in X, since f is SC

Therefore $X = \bigcup_{i=1}^{n} f^{-1}(0_i) = f^{-1}(\bigcup_{i=1}^{n} 0_i)$, since X is nearly compact.

Therefore $f(X) = f f^{-1}(\bigcup_{i=1}^{n} 0_i) \subset \bigcup_{i=1}^{n} 0_i$.

Hence $f(X)$ is compact.

Theorem 4.2 — If f is a SC, one-one mapping of X into Y and if X is nearly compact and Y is Hausdorff then f is almost open.

Proof : Let G be a regularly open subset of X. Then $X \sim G$ being a regularly closed subset of the nearly compact space X. Hence $X \sim G$ is nearly compact implies $f(X \sim G)$ is compact, since f is SC. Now as f is one-one, $f(X \sim G) = Y \sim f(G)$ is compact. Since Y is a Hausdorff space, $Y \sim f(G)$ is closed. Hence $f(G)$ is open implies f is almost open.

Definition 4.2 (Singal and Arya 1969) — A space X is said to be almost regular if for regularly closed set F and a point $x \notin F$ there exist disjoint open sets U and V such that $F \subseteq U$ and $x \in V$.

Theorem 4.3 — Let $f : X \to Y$ be a SC, one-one, almost open mapping of almost regular space X onto Y; then Y is regular space.

Proof : Let A be any closed subset of Y and $y \notin A$. Then since f is onto $f^{-1}(A) \cap f^{-1}(y) = \varphi$ and $f^{-1}(A)$ is δ-closed, since f is SC. Now as $f^{-1}(A)$ is δ-closed,

$f^{-1}(A) = \bigcap_{\alpha \in \Lambda} H_\alpha$, where H_α is a regularly closed set and $f^{-1}(A) \cap f^{-1}(y) = \varphi$. Therefore $\exists$ at least one $\alpha_0 \in \Lambda$ such that $H\alpha_0 \cap f^{-1}(y) = \varphi$ and $f^{-1}(A) \subset H\alpha_0$. Now since $H\alpha_0$ is a regularly closed set, $x \notin H\alpha_0$ and X is almost regular, $\exists$ disjoint open sets U and V such that $f^{-1}(A) \subset H\alpha_0 \subset U$ and $x \in V$. Now $U \cap V = \varphi$ implies $\bar{U}^i \cap V = \varphi$. Since f is almost open $A \subset f(\bar{U}^i)$ and $f(x) \in f(V)$ and $f(\bar{U}^i) \cap f(V) = \varphi$ since f is one-one. Hence Y is regular.

Definition 4.3 — A mapping $f : X \to Y$ is said to be an S-Homeomorphism if f is one-one and both f and f^{-1} are SC.

Definition 4.4 (Singal and Arya 1970) — A space $(X, \mathcal{T})$ is almost completely regular iff for every regularly (δ-closed Theorem 1) closed set A and a point $x \notin A$, there is a continuous function f on $(X, \mathcal{T})$ into the closed interval [0, 1] such that $f(x) = 1$ and $f(A) = 0$.

Theorem 4.4 — An S-Homeomorphic image of an almost completely regular space is completely regular.

PROOF: Let $(X, \mathcal{T})$ be an almost completely regular space and $f : X \to Y$ be an S-Homeomorphism. So there is a point x such that $x = f^{-1}(y)$. Also $x \notin f^{-1}(F)$ and $f^{-1}(F)$ is δ-closed for every closed set F in Y. Then $\exists$ a continuous mapping g : $X \to [0, 1]$ such that $g(x) = 0$ and $g[f^{-1}(F)] = 1$. Now let $h = g \circ f^{-1} : Y \to [0, 1]$. Then h is SC implies h is continuous. Now h is a continuous mapping from Y into [0, 1] such that $h(F) = 1$ and $h(x) = 0$. Hence the result.

REFERENCES

Singal, M. K., and Arya, Shashi Prabha (1969). On almost regular spaces. *Math. Vesni*, **6** (21), 1.

————(1970). Almost normal and almost completely regular spaces. *Glasnik Mat.*, **25**, 141–52.

Singal, M. K., and Mathur, Asha (1969). On nearly compact spaces. *Boll Un. Mat. Ital.* (4), **6**, 702–10.

————(1971). A note on almost regular spaces. *Glasnik Mat.*, **6**, 345–50.

————(1974). On nearly compact II. *Estra Ottoda* : *Boll. Un. Mat., Ital.* (4), **9**, 670–78.

Singal, M. K., and Singal, Asha Rani (1968). Almost continuous mapping. *Yokohama Math. J.*, **16**, 63–73.

Stone, M. H. (1977). Applications of the theory of boolean rings to General topology. *Trans. Am. Math. Soc.*, **41**, 375–481.

Velicko, N. V. (1966). *H*-Closed topological spaces. *Mat. Sab.* (*Russian*) (*N.S.*), **70** (112), 98–112.