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ALTERNATIVE METHODS OF OBTAINING DIFFERENTIAL OPERATOR REPRESENTATIONS

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Representation for the Jackson-Hermite polynomials in terms of a differential operator involving their generating function was obtained by Poli (1954) and for the Gegenbauer polynomials by Haradze (1964). Similar representations for the polynomials of Laguerre, Meixner and Poisson-Charlier were obtained recently by Allaway (1976). In this paper we give alternative methods for deriving such representations. Results obtained here include the Hermite, modified Jacobi, generalized Sylvester, Bessel, Konhauser biorthogonal, and Srivastava and Singhal polynomials. For the sake of brevity, the special cases already treated by Poli (1954), Haradze (1964) and Allaway (1976) have been omitted, although they can also be derived by our methods.

1. INTRODUCTION

Poli (1954) has obtained the following differential operator representation for the Jackson-Hermite polynomials $He_n(x)$:

$$yHe_n(x) = \sum_{k=0}^{n} \binom{n}{k} t^k y^{(n-k)}, \qquad \ldots(1.1)$$

where $\quad y^{(k)} = \dfrac{d^k y}{dt^k}$, and

$$y := \exp\,(xt - \tfrac{1}{2}t^2) = \sum_{n=0}^{\infty} He_n(x)\frac{t^n}{n!}\,.$$

A similar representation for the Gegenbauer polynomials $C_n^\alpha(x)$ was derived by Haradze (1964):

$$yC_n^\alpha(x) = \sum_{k=0}^{n} \binom{W}{k} \frac{t^k}{y^{(n-k)/\alpha}(n-k)!}\,\frac{d^{n-k}y}{dt^{n-k}}\,, \qquad \ldots(1.2)$$

where $\quad W = n + 2\alpha - 1$, and

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$$y := (1 - 2xt + t^2)^{-\alpha} = \sum_{k=0}^{\infty} C_k^{\alpha}(x)\, t^k.$$

Recently, following the technique used by Poli (1954) and Haradze (1964), Allaway (1976) obtained the following representations for the Laguerre, Meixner aud Poisson-Charlier polynomials:

$$yL_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{\alpha + n}{n - k} \frac{t^{n-k}(1 - t)^{2k} y^{(k)}}{k!} \qquad \ldots(1.3)$$

where

$$y := (1 - t)^{-1-\alpha} \exp\left(\frac{-xt}{1 - t}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n; \qquad \ldots(1.4)$$

$$m_n(x; \beta, c)\, Y = \frac{(\beta)_n}{c^n} \sum_{k=0}^{n} \binom{n}{k} \frac{t^{n-k}(c - t)^k (1 - t)^k}{(\beta)_k} (k)$$

where

$$Y := \left(1 - \frac{t}{c}\right)^{x} (1 - t)^{-x-\beta} = \sum_{n=0}^{\infty} m_n(x; \beta, c) \frac{t^n}{n!};$$

and

$$C_n(x; a)\, y = \sum_{k=0}^{n} \binom{n}{k} \left(1 + \frac{t}{a}\right)^k \left(\frac{t}{a}\right)^{n-k} y^{(k)} \qquad \ldots(1.5)$$

where

$$y := e^{-t}(1 + a^{-1}t)^{x} = \sum_{n=0}^{\infty} C_n(x; a) \frac{t^n}{n!}.$$

To obtain eqns. (1.1) to (1.5) Poli, Haradze and Allaway used the same technique which is now well known.

In this paper we give alternative methods of obtaining formulas of the aforementioned type for a variety of special functions. In section 2 we merely state some theorems (Theorems 2.1, 2.2, 2.3 and 2.4). The derivation of these theorems is either straightforward or is implied at once by known results. For example, Theorem 2.1 follows immediately by Equations (13) and (14) of Brafman (1959); see also the second step of the proof of Theorem 2 of Srivastava, Lavoie and Tremblay (1979, p. 11);

Theorem 2.3, on the other hand, is an obvious particular case of Theorem 1 of Srivastava (1971, p. 64). In the last section a number of applications of these general results are given. Representations for the Hermite, modified Jacobi, generalized Sylvester, Bessel, Konhauser biorthogonal, and Srivastava and Singhal polynomials are obtained. For the brevity of the paper, the special cases already treated by Poli (1954), Haradze (1964), and Allaway (1976) have been omitted. The importance of such representations, as pointed out by Haradze and Allaway, lies in the fact that we obtain linearly independent solutions of a class of homogeneous differential equations.

2. General Results

In this section we state four theorems of general nature which yield formulas of the aforementioned type for many special functions. [See also the remarks about these theorems in the preceding section.]

Theorem 2.1 — Let the sequence of the functions $\{S_n(x) \mid n = 0, 1, 2, \ldots\}$ be generated by

$$F[x, t] = \sum_{n=0}^{\infty} S_n(x)\, t^n. \qquad \ldots(2.1)$$

Further, assume that the series involved in eqn. (2.1) is uniformly convergent; then

$$\frac{\partial^k F}{\partial t^k} = k! \sum_{n=0}^{\infty} \binom{n+k}{k} S_{n+k}(x)\, t^n \qquad \ldots(2.2)$$

where k is a nonnegative integer.

Theorem 2.2 — Let the sequence of the functions $\{f_n(x)\ n = 0, 1, \ldots\}$ be generated by

$$A(t)\, f(x, t) = \sum_{n=0}^{\infty} f_n(x)\, t^n \qquad \ldots(2.3)$$

where $A(t)$ is an arbitrary function of t and $f(x, t)$ is an arbitrary function of t and $f(x, t)$ is an arbitrary function of x and t.

Then

$$f_n(x, y) = \sum_{k=0}^{n} g_k(x, y)\, f_{n-k}(x) \qquad \ldots(2.4)$$

where the functions $g_n(x, y)$ are defined by the generating relation

$$f(xy, t)\, \{f(x, t)\}^{-1} = \sum_{n=0}^{\infty} g_n(x, y)\, t^n. \qquad \ldots(2.5)$$

The following theorem which is a particular case of the Theorem 2.2 is of special significance as it can be applied directly in many cases.

Theorem 2.3 — Let the sequence of functions $\{f_n(x, a; r) \mid n = 0, 1, 2, ...\}$ be defined by the generating relation

$$e^{at^r} f(xt) = \sum_{n=0}^{\infty} f_n(x, a, r)\, t^n \quad ...(2.6)$$

where $a \neq 0$ is an arbitrary constant, r is an arbitrary positive integer and $f(u)$ is an arbitrary function of u. Then

$$f_n(x\lambda, a, r) = \sum_{k=0}^{[n/r]} \lambda^{n-kr}\{a(1-\lambda)^r\}\, f_{n-kr}(x, a, r) \quad ...(2.7)$$

where $\lambda \neq 0$ is an arbitrary constant.

Theorem 2.4 — Let the sequence of functions $\{g_n(x)\ n = 0, 1, 2, ...\}$ be defined by the generating relation

$$e^{bxt} A(t) = \sum_{n=0}^{\infty} g_n(x)\, t^n \quad ...(2.8)$$

where $b \neq 0$ is an arbitrary constant and $A(t)$ is an arbitrary power series in t. Then

$$g_n(x + \mu) = \sum_{k=0}^{n} \frac{(b\mu)^k}{k!} g_{n-k}(x) \quad ...(2.9)$$

μ being a constant.

3. Representations

We have developed everything that is required to find the differential operator representations for certain special functions. We discuss here the following particular cases:

(I) *Hermite Polynomials*

The Hermite polynomials $H_n(x)$ are defined by means of the relation (Rainville 1960, eqn. (1), p. 187)

$$y_1 := \exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{n!}. \quad ...(3.1)$$

According to Rainville (1960, eqn. (1), p. 197), the Hermite polynomials also satisfy the relation

$$\sum_{n=0}^{\infty} \frac{H_{n+k}(x)\, t^n}{n!} = \exp(2xt - t^2)\, H_k(x - t), \qquad \ldots(3.2)$$

and so Theorem 2.1, when applied to the Hermite polynomials yields

$$y_1^{(k)} = y_1 H_k(x - t). \qquad \ldots(3.3)$$

Next, on applying Theorem 2.4 to eqn. (3.1) we get

$$H_n(x + \mu) = \sum_{k=0}^{n} \binom{n}{k} (2\mu)^k H_{n-k}(x). \qquad \ldots(3.4)$$

Using (3.3) in (3.4) [with $x \to x - t$ and $\mu = t$], we obtain the following differential operator representation for the Hermite polynomials:

$$y_1 H_n(x) = \sum_{k=0}^{n} \binom{n}{k} (2t)^k y_1^k \qquad \ldots(3.5)$$

where y_1 is defined by (3.1).

It is worthwhile to remark here that Representation (1.1) can be deduced from (3.5) in view of the relations

$$He_n(x) = 2^{-n/2} H_n[x/\sqrt{2}],\ H_n(x) = 2^{n/2} He_n(x\sqrt{2}).$$

(II) *Modified Jacobi Polynomials*

According to Carlitz (1963, p. 88) the modified Jacobi polynomials are generated by

$$y_2 := \sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x)\, t^n$$

$$= [1 + \tfrac{1}{2}(x + 1)t]^{\alpha} [1 + \tfrac{1}{2}(x - 1)\, t]^{\beta}. \qquad \ldots(3.6)$$

Recalling the following known formula (see Manocha and Sharma 1966, p. 460)

$$\sum_{n=0}^{\infty} \binom{n + k}{k} P_{n+k}^{(\alpha-k-n,\beta-k-n)}(x)\, t^n$$

$$= \{1 + \tfrac{1}{2}(x + 1)\, t\}^{\alpha-k} \{1 + \tfrac{1}{2}(x - 1)\, t\}^{\beta-k} \cdot P_k^{(\alpha-k,\beta-k)}(x + \tfrac{1}{2}(x^2 - 1)\, t) \qquad \ldots(3.7)$$

and applying our Theorem 2.1 to the modified Jacobi polynomials, we obtain

$$y_2^{(k)} = k!\{1 + \tfrac{1}{2}(x+1)\,t\}^{-k}\,\{1 + \tfrac{1}{2}(x-1)\,t\}^{-k}\cdot y_2 P_k^{(\alpha-k,\beta-k)}\,(Z_2) \quad \ldots(3.8)$$

where $Z_2 = (x + \tfrac{1}{2}(x^2 - 1)\,t)$.

Next, taking $a = 1 = r$ and

$$f(xt) = {}_{p+1}F_q \left[\begin{matrix} -n, \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{matrix} \; -xt \right],$$

we obtain from Theorem 2.3 the following formula (see Luke 1969; eqn. 27, p. 11):

$${}_{p+1}F_q \left[\begin{matrix} -n, \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{matrix} \; x \right]$$

$$= \sum_{k=0}^{n} \binom{n}{k} \lambda^k (1-\lambda)^{n-k}\, {}_{p+1}F_q \left[\begin{matrix} -k, \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{matrix} \; x \right]. \quad \ldots(3.9)$$

Combining eqn. (3.8) and eqn. (3.9), we arrive at

$$y_2 P_n^{(\alpha-n,\beta-n)}(x) = \sum_{k=0}^{n} \binom{-1-\alpha-\beta+n}{n-k} \cdot [\tfrac{1}{4}t(x^2-1)]^{n-k}$$

$$\times [\{1 + \tfrac{1}{2}t(x+1)\}\,\{1 + \tfrac{1}{2}t(x-1)\}]^k \cdot \frac{y_2^{(k)}}{k!}$$

$\ldots(3.10)$

where y_2 is given by eqn. (3.6).

(III) *Generalized Sylvester Polynomials*

Define the polynomials $\{f_n(x; a) \mid n = 0, 1, 2, \ldots\}$ where $a \neq 0$ is a constant, by means of the relation

$$y_3 := \sum_{n=0}^{\infty} f_n(x; a)\, t^n = (1-t)^{-x}\, e^{axt}. \quad \ldots(3.11)$$

We call the polynomials $\{f_n(x; a) \mid n = 0, 1, 2, \ldots\}$ as generalized Sylvester polynomials in view of the relation

$$f_n(x; 1) = \phi_n(x), \quad \ldots(3.12)$$

where $\phi_n(x)$ is the Sylvester polynomial of degree n. (see Rainville 1960, p. 302). The explicit expression for the polynomials $f_n(x; a)$ is

$$f_n(x; a) = \frac{(ax)^n}{n!}\, {}_2F_0[-n, x; -; -(1/ax)]. \quad \ldots(3.13)$$

Replacing t by $t + u$ in eqn. (3.11), we get

$$\sum_{n,\, k=0}^{\infty} \binom{n+k}{k} f_{n+k}(x; a)\, t^n u^k$$

$$= (1 - t)^{-x} e^{axt} \left(1 - \frac{u}{1-t}\right)^{-x} e^{a(1-t)x} \frac{u}{(1-t)}$$

$$= (1 - t)^{-x}\, e^{axt} \sum_{k=0}^{\infty} f(x; a(1 - t)) \left(\frac{u}{1-t}\right)^k .$$

By equating coefficients of u^k, we get

$$\sum_{n=0}^{\infty} \binom{n+k}{k} f_{n+k}(x; a) t^n = (1 - t)^{-x-k} e^{axt} f(x; a(1 - t)). \qquad \ldots(3.14)$$

Applying our Theorem 2.1 to the generalized Sylvester polynomials and using eqn. (3.14), we get

$$y_3^{(k)} = k!\, (1 - t)^{-k}\, y_3 f_k(x; a(1 - t)). \qquad \ldots(3.15)$$

Following the method of proof of the Theorem 2.2, one may easily prove that

$$f_n(x; ab) = \sum_{k=0}^{n} \frac{\{ax(b-1)\}^{n-k}}{(n-k)!} f_k(x; a), \qquad \ldots(3.16)$$

$b \neq 0$ being a constant.

Equation (3.16) gives

$$f_n(x; a) = \sum_{k=0}^{n} \frac{(axt)^{n-k}}{(n-k)!} f_k(x; a(1 - t)). \qquad \ldots(3.17)$$

Using eqn. (3.15) in eqn. (3.17), we get the following differential operator representation for the polynomials $f_n(x; a)$:

$$y_3 f_n(x; a) = \sum_{k=0}^{n} \frac{(axt)^{n-k}(1-t)^k}{k!(n-k)!}\, y_3^{(k)}. \qquad \ldots(3.18)$$

(IV) *Bessel Polynomials*

According to Rainville [1960, eqn. (2), p. 294] the generalized Bessel polynomial $y_n(x)$ is defined by the explicit expression

$$y_n(a, b, x) = {}_2F_0(-n, a - 1 + n; -; -(x/b)). \qquad \ldots(3.19)$$

In the present sub-section we shall obtain differential operator representation for the polynomial $y_n(a - n, b, x)$ which has the generating relation

$$y_4 := \sum_{n=0}^{\infty} y_n(a - n, b, x)\,(t^n/n!)$$

$$= (1 - (xt/b))^{1-a}\, e^t. \qquad \ldots(3.20)$$

The polynomial $y_n(a - n, b, x)$ also satisfies the generating relation (Srivastava and Lavoie 1975, eqn. (54), p. 311)

$$\sum_{n=0}^{\infty} y_{n+k}(a - n - k, b, x)\,\frac{t^n}{n!}$$

$$= \left(1 - \frac{xt}{b}\right)^{1-a} e^t y_k\left(a - k, b, x\left(1 - \frac{xt}{b}\right)^{-1}\right). \qquad \ldots(3.21)$$

Applying Theorem 2.1 to the polynomials $y_n(a - n, b, x)$ and using eqn. (3.21), we get

$$y_4^{(k)} = y_4 y_k\left(a - k, b, x\left(1 - \frac{xt}{b}\right)^{-1}\right). \qquad \ldots(3.22)$$

Using eqn. (3.9), we get

$$y_n(a - n, b, x) = \sum_{k=0}^{n} \binom{n}{k}\left(1 - \frac{xt}{b}\right)^{k}\left(\frac{xt}{b}\right)^{n-k}$$

$$\times\ y_k\left(a - k, b, x\left(1 - \frac{xt}{b}\right)^{-1}\right). \qquad \ldots(3.23)$$

Using eqn. (3.22) in eqn. (3.23), we obtain

$$y_4 y_n(a - n, b, x) = \sum_{k=0}^{n} \binom{n}{k}\left(1 - \frac{xt}{b}\right)^{k}\left(\frac{xt}{b}\right)^{n-k} y_4^{k}. \qquad \ldots(3.24)$$

(V) *Srivastava and Singhal Polynomials*

The Srivastava and Singhal polynomials are defined by the generating function (Srivastava and Singhal 1971, eqn. (3.2), p. 78)

$$y_5 := (1 - kt)^{-\alpha/k} \exp\left[px^r\{1 - (1 - kt)^{-r/k}\}\right]$$

$$= \sum_{n=0}^{\infty} G_n^{(\alpha)}(x, r, p, k)\ t^m. \qquad \ldots(3.25)$$

Srivastava and Singhal (1971, eqn. (5.6), p. 79) have also proved the following generating relation

$$(1 - kt)^{-m-\alpha/k} \exp [px^r\{1 - (1 - kt)^{-r/k}\}] \cdot G_m^{(\alpha)} (x(1 - xt)^{-1/k}, r, p, k)$$

$$= \sum_{n=0}^{\infty} \binom{m+n}{n} G_{m+n}^{(\alpha)} (x, r, p, x)\, t^n. \qquad \ldots(3.26)$$

Applying Theorem 2.1 to the polynomials $G_n^{(\alpha)}(x, r, p, k)$, we obtain

$$y_5^{(m)} = m!(1 - t)^{-m}\, y_5 G_m^{(\alpha)} (x(1 - kt)^{-1/k}, r, p, k). \qquad \ldots(3.27)$$

Using Theorem 2.2, we get

$$G_n^{(\alpha)} (x, r, p, k) = \sum_{m=0}^{n} f_{n-m}(x(1 - kt)^{-1/k}, (1 - kt)^{1/k})$$

$$\times\, G_m^{(\alpha)} (x(1 - kt)^{-1/k}, r, p, k), \qquad \ldots(3.28)$$

where the functions $\{f_n(x, y) \mid n = 0, 1, 2, \ldots\}$ are defined by the generating function

$$\exp [px^r(y^r - 1) \{1 - (1 - kt)^{-r/k}\}] = \sum_{n=0}^{\infty} f_n(x, y)\, t^n.$$

Using eqn. (3.27) in eqn. (3.28), we get the following representation for the polynomials $G_n^{(\alpha)} (x, r, p, k)$:

$$y_5 G_n^{(\alpha)} (x, r, p, k) \qquad \ldots(3.29)$$

$$= \sum_{m=0}^{n} (1 - t)^m f_{n-m}(x(1 - kt)^{-1/k}, (1 - kt)^{1/k}) \frac{y_5^{(m)}}{m!}. \qquad \ldots(3.30)$$

For the Konhauser bi-orthogonal polynomials $Y_n^{\alpha}(x; k)$, where $\alpha > -1$ and k is a positive integer, the following relationship is known (see Srivastava and Lavoie 1975, eqn. (83), p. 315):

$$Y_n^{\alpha}(x; k) = k^{-n} G_n^{(\alpha+1)} (x, 1, 1, k), \qquad \ldots(3.31)$$

From (3.30) and (3.31), it is easy to obtain the following differential operator representation for the polynomials $Y_n^{\alpha}(x; k)$:

$$y Y_n^{\alpha} (x; k) = \sum_{m=0}^{n} \frac{(1 - t)^m g_{n-m}(x(1 - t)^{-1/k}, (1 - t)^{1/k})\, y^{(m)}}{m!}, \qquad \ldots(3.32)$$

where

$$y = \sum_{n=0}^{\infty} Y_n^{\alpha}(x; k)\, t^n$$

$$= (1 - t)^{-(\alpha+1)/k} \exp [x\{1-(1 - t)^{-/k}\}], \qquad ...(3.33)$$

and the functions $\{g_n(x, y) \mid n = 0, 1, 2, ...\}$ are defined by the generating function

$$\exp [x(y - 1) \{1 - (1 - t)^{-1/k}\}]$$

$$= \sum_{n=0}^{\infty} g_n(x, y)\, t^n. \qquad ...(3.34)$$

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