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ON SOME SUBCLASSES OF UNIVALENT FUNCTIONS REPRESENTED BY INTEGRAL

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Let $f(z)$ and $g(z)$ be normalised analytic functions. For $\alpha > 0$, $-\pi/2 < \theta < \pi/2$ and Re $c \geqslant 0$, let

$$F(z) = [(\alpha+c)z^{-c} \int_0^z f^{\alpha}(t)\, t^{c-1}\, dt]^{1/\alpha}$$

and $$H(z) = [(\alpha+c)z^{-c_2} \int_0^z (g(t))^{\alpha(1+i \tan\theta)}\, t^{c_2-1}\, dt]^{1/\alpha(1+i \tan\theta)}$$

where $c_2 = c - i\alpha \tan\theta$. It is proved that if $f(z)$ is starlike of order ρ then so is $F(z)$ and if $g(z)$ is θ-spiral-like of order ρ then so is $H(z)$. Hardy classes for the starlike function $F(z)$ and the spiral-like function $H(z)$ are determined.

Let $g(z)$ be analytic in the unit disc $E = \{z : |z| < 1\}$ and θ be a real number such that $|\theta| < \pi/2$. If $g(0) = 0$, $g'(0) \neq 0$ and Re $[e^{i\theta} zg'(z)/g(z)] > 0$ for z in E, then $g(z)$ is univalent Špaček (1933) and is said to be θ-spiral-like (Libera 1967). Under these conditions we have

$$zg'(z)/g(z) = e^{-i\theta} [\cos\theta P(z) + i \sin\theta] \qquad ...(1)$$

where Re $P(z) > 0$ in E. Further, if $g'(0) = 1$ (i.e. $P(0) = 1$) and if in (1) Re $P(z) \geqslant \rho$, $0 \leqslant \rho < \cos\theta$. We shall say that $g(z)$ is in $F_{\theta}(\rho)$. It is clear from the definition that $\bigcup_{0 \leqslant \rho \leqslant \cos\theta} F_{\theta}(\rho) = F_{\theta}(0) \equiv F_{\theta}$, the whole class of spiral-like functions. In particular with $\alpha = 0$, $F_0(\rho)$ is the class $S^*(\rho)$ of normalised starlike functions of order ρ, $F_0(0)$ being the class S^* of all normalised starlike functions.

We say that an operator is a spiral-like operator, if it is defined on F_{θ}, and maps F_{θ} into (or onto) F_{θ}. A fortiori, an operator is a starlike operator if it is defined on S^* and maps S^* into (or onto) S^*. Consider the integral operator

$$F(z) = (Tf)(z) = [(\alpha+c)\, z^{-c} \int_0^z f^{\alpha}(t)\, t^{c-1}\, dt]^{1/\alpha}. \qquad ...(2)$$

Recently Ruscheweyh (1973, Theorem 3.2) has shown that T is a starlike operator when $\alpha > 0$ and Re $c \geqslant 0$.

Finally, for $\lambda > 0$, we say that a function $h(z)$ analytic in E belongs to the Hardy class H^{λ} if $\lim_{r \to 1-} \int_{-\pi}^{\pi} | f(re^{i\theta}) |^{\lambda} \, d\theta$ exists and is finite.

In this note, we first extend the above result of Ruscheweyh and prove that T maps $S^*(\rho)$ into $S^*(\rho)$ $(0 \leqslant \rho < 1)$. With the help of the operator T we study a corresponding spiral-like operator T_θ (to be defined latter). We determine the Hardy class to which functions in the classes $T(S^*(\rho))$ and $T_\theta(F_\theta(\rho))$ belong. Our results generalise the Hardy class results given by Eenigenburg *et al.* (1973, 1974).

We first state some known results which we will need in the proof our results.

Theorem A (Basgöze and Keogh 1970) — A function $g(z)$ is in $F_\theta(\rho)$ if and only if there exists $f(z)$ in $S^*(\rho)$ such that

$$g(z) = z \, [f(z)/z]^{1/(1+i \tan \theta)} \qquad \ldots(3)$$

where the branch is choosen so that $[f(z/z)]^{1/(1+i \tan \theta)} = 1$ at $z = 0$.

Theorem B — If $P(z)$ is analytic and Re $P(z) > 0$ in E then $P(z)$ is in H^{λ} for $\lambda < 1$.

Theorem C (Eenigenburg 1970) — If $f(z)$ is in $S^*(\rho)$ and is not of the form $f(z) = \frac{z}{(1 - ze^{it})^{2(1-\rho)}}$ for some real t then

(i) there exists $\epsilon = \epsilon(f) > 0$ such that $(g(z)/z)$ is in $H^{(1/2(1-\rho))+\epsilon}$

(ii) there exists $\epsilon = \epsilon(f) > 0$ such that $g'(z)$ is in $H^{(1/3-2\rho)+\epsilon}$

Theorem B can be found in any standard texts.

We now prove the following results.

Lemma 1 — The operator T defined by (2) maps $S^*(\rho)$ into $S^*(\rho)$, when $\alpha > 0$ and Re $c \geqslant 0$.

PROOF : A function $f(z)$ is in $S^*(\rho)$ if and only if there exists a function $s(z)$ in S^* such that $f(z) = z \, [s(z)]^{(1-\rho)}$. A simple calculation shows that

$$F(z) = (Tf)(z) = z \, [(TS) \, (z)]^{(1-\rho)} = z \, [G(z)]^{1-\rho}$$

where $G(z)$ is in S^* by Theorem 3.2 of Ruscheweyh (1973). Thus, the theorem is proved.

Lemma 2 — Let c be a complex number with nonnegative real part and α and θ are real numbers such that $\alpha > 0$ and $| \theta | < \pi/2$. Then the operator T_θ defined on $F_\theta(\rho)$ by the formula

$$H(z) = (T_\theta g)(z) = [(c+\alpha)z^{-c_2} \int_0^z (g(t))^{\alpha(1+i \tan \theta)} \cdot t^{c_2-1}\, dt]^{1/(\alpha(1+i \tan \theta))} \qquad \ldots(4)$$

is a spiral-like operator and maps $F_\theta(\rho)$ into $F_\theta(\rho)$, where $c_2 = c - i\alpha \tan \theta$.

PROOF : Let the function $f(z)$ in $S^*(\rho)$ be defined by the formula (3). By Lemma 1, the function $F(z) = (Tf)(z)$ defined by (2) is in $S^*(\rho)$. Since $H(z) = (T_\theta g)(z) = \left[\frac{F(z)}{z}\right]^{1/(1+i \tan \theta)}$ it follows, by Theorem A, that $H(z)$ is in $F_\theta(\rho)$. This completes the proof.

Let $F_\alpha(z)$ denote the function obtained by replacing $f(z)$ in (2) by the function $K(z) = z/(1-z)^{2(1-\rho)}$.

Theorem 1 — Let $F(z)$ be defined by (2) where $f(z)$ is in $S^*(\rho)$, $\alpha > 0$ and Re $c \geqslant 0$.

(i) If $0 < \alpha \leqslant \frac{1}{2(1-\rho)}$, then $F(z)$ is bounded unless it is a rotation or magnitification of $F_{1/(2(1-\rho))}(z)$.

(ii) If $\alpha > \frac{1}{2(1-\rho)}$ and $F(z)$ is not a rotation or magnification of $F_\alpha(z)$ then, there exists $\epsilon = \epsilon(F) > 0$ such that $F(z)$ is in $H^{(\alpha/2(1-\rho)\alpha-1)+\epsilon}$ and $F'(z)$ is in $H^{(\alpha/(3-2\rho)\alpha-1)+\epsilon}$.

(iii) For $\alpha > \frac{1}{2(1-\rho)}$, $F_{1/(2(1-\rho))}(z)$ is in H^p for all $p < \frac{\alpha}{2(1-\rho)\alpha-1}$ but does not belong to $H^{\alpha/(2(1-\rho)\alpha-1)}$.

PROOF : We define

$$q(z) = (f(z)/z)^\alpha = \sum_{n=0}^{\infty} c_n z^n. \qquad \ldots(5)$$

Since $f(z)/z \neq 0$, a single valued analytic branch of $q(z)$ is well defined. If we write

$$G(z) = \sum_{n=0}^{\infty} \frac{c_n(\alpha+c)}{(n+\alpha+c)} z^n \qquad \ldots(6)$$

then $G(z)$ is analytic in $|z| < 1$. Ruscheweyh (1973, Theorem 3.2) has shown that $G(z) \neq 0$ and

$$F(z) = z(G(z))^{1/\alpha}. \qquad \ldots(7)$$

Further, $G(z)$ satisfies

$$G(z) + \frac{zG'(z)}{(\alpha + c)} = q(z).$$

Hence by (7) and (5)

$$\frac{zG'(z)}{(\alpha + c)} = (f(z)/z)^{\alpha} - (F(z)/z)^{\alpha}. \qquad ...(8)$$

From (5) and (6), after a brief calculation, we can see that $F(z)$ cannot be a rotation or magnification of $z/(1-z)^{2(1-\rho)}$. Let $f(z)$ also be not a rotation or magnification of $z/(1-z)^{2(1-\rho)}$. Thus, from Theorem C and (8) it follows that

$G'(z)$ is in $H^{1/(2(1-\rho)\alpha)+\epsilon}$.

Now for $0 < \alpha \leqslant \frac{1}{2(1-\rho)}$, $G(z)$ is bounded (Duren 1970, p. 91). Hence by the relation (7) $F(z)$ is also bounded. For $\alpha > \frac{1}{2(1-\rho)}$, we use a result due to Hardy and Littlewood (Duren 1970, p. 88) and it follows that $G(z)$ is in $H^{(1/(2(1-\rho)\alpha-1))+\epsilon}$. Thus, by (7), $F(z)$ is in

$H^{(\alpha/(2(1-\rho)\alpha-1))+\epsilon}$ (ϵ possibly different). ...(9)

Next, we show that $F'(z)$ is in $H^{(\alpha/(3-2\rho)\alpha-1))+\epsilon}$. By relation (1), $F'(z) = F(z)P(z)/z$ where Re $P(z) > 0$. We take ϵ defined in (9) and choose δ so small that

$$\epsilon > \delta(\lambda+\epsilon)\lambda, \text{ where } \lambda = \alpha/2(1-\rho)\alpha - 1. \qquad ...(10)$$

Now write $p = \frac{\lambda+\epsilon}{K}$, $q = \frac{1}{K(1+\delta)}$ where $K = (\lambda+\epsilon)/(1+\lambda+\epsilon+\delta\lambda+\delta\epsilon)$. With such a choice of K, p and q are conjugate indices in the Holder's inequality. Thus,

$$\int_{-\pi}^{\pi} |F'(z)|^K \, d\theta \leqslant \left(\int_{-\pi}^{\pi} |F(z)/z|^{Kp} \, d\theta\right)^{1/p} \left(\int_{-\pi}^{\pi} |P(z)|^{Kq} \, d\theta\right)^{1/q}$$

$z = re^{i\theta}$. By (9) and by Theorem B, it follows that $\lim_{r\to 1-} \int_{-\pi}^{\pi} |F'(z)|^K \, d\theta$ is finite. By (10), $K > \frac{\lambda}{1+\lambda}$. Hence there exists $\epsilon = \epsilon(F) > 0$ such that $F'(z)$ is in $H^{(\alpha/(3-2\rho)\alpha-1)+\epsilon}$.

We use (8) and Theorem C to verify part (iii). This completes the proof.

Remark 1 : We note that if $f(z)$ is a convex function in (2) and $0 < \alpha \leqslant 1$, then $F(z)$ is bounded.

Remark 2 : When $\beta = 1/\alpha > 0$, $c = 0$, (2) is a representation for β-convex functions (Miller *et al.* 1973). The bounds for the Hardy class obtained in Theorem 1 when $\rho = 0$ are precisely those obtained by Eenigenburg and Miller (1973) for β-convex functions.

Theorem 2 — Let $g(z)$ and $H(z)$ be the functions in $F_\theta(\rho)$ defined in Lemma 2 and $f(z)$ be the function defined by the relation (3)

(i) If $0 < \alpha < \dfrac{1}{2(1-\rho)}$, then $H(z)$ is bounded unless $f(z) = z/(1 - e^{it} z)^{2(1-\rho)}$ where t is a real number.

(ii) If $\alpha > \frac{1}{2}$ and $f(z) \neq z/(1 - e^{it} z)^{2(1-\rho)}$ then there exists $\epsilon = \epsilon(H) > 0$ such that $F(z)$ is in $H^{\lambda+\epsilon}$ where $\lambda = \dfrac{\alpha \sec^2 \theta}{2(1-\rho)\alpha - 1}$ and $F'(z)$ is in $H^{(\lambda/1+\lambda)+\epsilon}$.

(iii) For $\alpha > \dfrac{1}{2(1-\rho)}$, the function $H(z)$ obtained in (4) by taking $g(z) = z\,[(1-z)^{-2(1-\rho)}]^{1/(1+i\tan\theta)}$ belongs to H^p for all $p < \lambda$ but does not belong to H^λ.

PROOF : From the proof of Lemma 2, we see that

$$\frac{H(z)}{z} = \left[\frac{F(z)}{z}\right]^{(\cos^2\theta - i\sin\theta\cdot\cos\theta)}$$

where $F(z)$ defined by (2) is in $S^*(\rho)$. Thus,

$$\left|\frac{H(z)}{z}\right|^{\sec^2\theta} = \left|\frac{F(z)}{z}\right| \exp\left(\tan\theta \arg\left(\frac{F(z)}{z}\right)\right). \qquad \ldots(11)$$

The second term in the right-hand side of (11) is bounded. This, together with Theorem 1, determines the Hardy class for $H(z)$. For the derivative a proof similar to Theorem 1 can be easily constructed. This completes the proof.

Remark 3 : When $c = i\alpha\tan\theta$ and $\rho = 0$, (4) is a representation for a class of Bazilevic functions $B(\alpha,\beta)$ studied by Eenigenburgh *et al.* (1974). When $c = 0$ and $\rho = 0$, (4) is a representation for a class of spiral-like functions generated from $\dfrac{1}{\alpha}$ – convex functions (Miller *et al.* 1973) by the formula (3). Except for notation, the bounds for the Hardy class obtained in Theorem 2 are precisely those obtained for the special class of Bazilevic functions $B(\alpha, \beta)$ by Libera (1967).

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