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STRUCTURE THEOREMS FOR *H*-TRANSFORMABLE GENERALIZED FUNCTIONS

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In this paper we have discussed the representation of H-transformable generalized functions interms of differential operators acting on functions or on measures.

1. INTRODUCTION

Gel'fand and Shilov (1968, pp. 109–121), Koh (1970) and Pandey (1971, 1972) have investigated the representation of different kinds of generalized functions. Here we obtain the structure theorems for the H-transformable generalized functions. The H-transform of a suitably restricted function $f(t)$ is defined as

$$F(s) \triangleq \int_0^\infty \lambda(st)\, f(t)\, dt \qquad \ldots(1.1)$$

where $$\lambda(st) \triangleq e^{-wst}\, H^{m,n}_{p,q}\left[\rho(st)^{\mu} \,\middle|\, \begin{matrix} \{(a_p, A_p)\} \\ \{(b_q, B_q)\} \end{matrix} \right] \qquad \ldots(1.2)$$

and $H^{m,n}_{p,q}(z)$ is Fox's H-function (1961) which is defined as Mellin-Barnes type integral

$$H^{m,n}_{p,q}(z) = \frac{1}{2\pi i} \int_L X(s)\, z^s ds \qquad \ldots(1.3)$$

for a suitable contour L and

$$X(s) = \frac{\prod_1^m \Gamma(b_j - B_j s) \prod_1^n \Gamma(1 - a_j + A_j s)}{\prod_{m+1}^q \Gamma(1 - b_j + B_j s) \prod_{n+1}^p \Gamma(a_j - A_j s)} \qquad \ldots(1.4)$$

The asymptotic behaviour of H-function is discussed in detail by Braaksma (1963).

2. THE TESTING FUNCTION SPACES $H_{a,b}$, $H(a, b)$ AND THEIR DUALS

We say that a smooth function $\phi(t)$ defined on $I = (0, \infty)$ belongs to $H_{a,b}$ if

$$\gamma_k(\phi) = \sup_{0<x<\infty} \left| e^{-bx} x^{a+k} D^k \phi(x) \right| < \infty \qquad \ldots(2.1)$$

for $k = 0, 1, 2, \ldots$; a, b are fixed real numbers. The collection of semi-norms $\{\gamma_k\}_{k=0}^{\infty}$ is separating. $H_{a,b}$ is complete countably multinormed space. It can be shown that $H_{a,b}$ is a testing function space (Joshi and Saxena 1981). Therefore the dual $H'_{a,b}$ of $H_{a,b}$ is a space of generalized functions. If $0 < b_1 < b_2$ then H_{a,b_1} is a subspace of H_{a,b_2}. By using this property, we now construct the countable union space, $H(a, b)$. Let b denote a real number or ∞. Choose a monotonic increasing sequence $\{b_v\}_{v=1}^{\infty}$ of real positive numbers such that $b_v \to b^-$. Then

$$H(a, b) = \bigcup_{v=1}^{\infty} H_{a,b_v} \qquad \ldots(2.2)$$

is the countable union space (Joshi and Saxena 1980). Hence $H'(a, b)$ is a space of generalized functions.

3. The Generalized H-Transformation

We shall call generalized function f a H-transformable if it belongs to $H'(a, b)$, for some real numbers a and b. Let

$$\sigma_f = \inf \{b \mid f \in H'(a, b)\} \qquad \ldots(3.1)$$

and

$$\Omega_f = \left\{s : \text{Re } s > -\frac{\sigma_f}{w}\right\}. \qquad \ldots(3.2)$$

We define H-transform

$$(\mathcal{H} f)(s) \triangleq F(s) \triangleq \langle f(t), \lambda(st) \rangle \qquad \ldots(3.3)$$

of f as an application of f belonging to $H'(a, b)$ to the function $\lambda(st)$ given by (1.2) which is an element of $H(a, b)$.

4. The Space $\bar{H}_{a,b}$

A complex valued smooth function $\phi(x)$ defined on I is said to be a member of $\bar{H}_{a,b}$ if it satisfies the following order properties

$$\begin{aligned} D^k\phi(x) &= O(1),\ x \to 0 \\ &= O(1),\ x \to \infty. \end{aligned} \qquad \ldots(4.1)$$

Lemma 1 — $\bar{H}_{a,b}$ is a linear subspace of $H_{a,b}$ where $a \geqslant 0$ and $b > 0$.

PROOF : The proof of Lemma 1 is very simple and hence omitted.

Lemma 2 — Let $\bar{H}_{a,b}$ the space of all smooth functions ϕ defined on $I = (0, \infty)$ satisfying the order properties (4.1). Then for $\phi \in \bar{H}_{a,b}$ and $f \in H'_{a,b}$ there exist a positive constant C and a nonnegative integer r such that

$$| \langle f, \phi \rangle | \leqslant C \max_{1 \leqslant k \leqslant r+1} \int_0^\infty | P_{a,b}(x)\, x^{k-1} D^k \phi(x) |\, dx \quad \ldots(4.2)$$

where

$$P_{a,b}(x) = e^{-bx} x^a. \quad \ldots(4.3)$$

PROOF : The proof of Lemma 2 follows from the boundedness property of generalized functions and elementary rules of integral calculus.

Now we are in position to state and prove the main structure theorem.

Theorem 1 — Let $f \in H'_{a,b}$ and $\phi \in \bar{H}_{a,b}$, then there exist M bounded measurable functions $g_k(x)$, $1 \leqslant k \leqslant r+1$ defined on $L_\infty(0, \infty)$ such that

$$| \langle f, \phi \rangle | = \sum_{k=1}^{r+1} \langle g_k(x),\ P_{a,b}(x)\, x^{k-1} D^k \phi(x) \rangle \quad \ldots(4.4)$$

where $P_{a,b}(x)$ is given by (4.3).

PROOF : On account of Lemma 2 we have

$$| \langle f, \phi \rangle | \leqslant C \max_{1 \leqslant k \leqslant r+1} \int_0^\infty | P_{a,b}(x)\, x^{k-1} D^k \phi(x) |\, dx$$

$$\therefore | \langle f, \phi \rangle | \leqslant C \max_{1 \leqslant k \leqslant r+1} \| P_{a,b}(x)\, x^{k-1} D^k \phi(x) \|_{L_1(0,\infty)} \quad \ldots(4.5)$$

where $L_1(0, \infty)$ is the space of all equivalence classes of Lebesgue integrable functions on $(0, \infty)$ whose topology is defined through the norm

$$\| \psi(x) \| = \int_0^\infty | \psi(x) |\, dx < \infty. \quad \ldots(4.6)$$

The result (4.5) defines a linear one-to-one and into mapping

$$M : \bar{H}_{a,b} \to L_1(0, \infty)$$

as $$\phi \mapsto P_{a,b}(x)\, x^{k-1} D^k \phi(x), \quad 1 \leqslant k \leqslant r+1.$$

Since $\bar{H}_{a,b}$ is a linear subspace of $L_1(0, \infty)$, (4.5) further states that f is continuous linear functional (Zemanian 1968, p. 26) on $\bar{H}_{a,b}$ in the topology induced on it by $L_1(0, \infty)$. Hence by Hahn-Banach theorem (Riesz and Nagy 1956, p. 114) f can be extended as continuous linear functional on the whole of $L_1(0, \infty)$. But the conjugate of $L_1(0, \infty)$, the space of all bounded measurable functions on $(0, \infty)$ such that for every $f \in L_\infty(0, \infty)$ there exists M such that $| f | \leqslant M$ almost everywhere.

Therefore on account of Reisz-representation theorem (Riesz and Nagy 1956, p. 78) there exist M-bounded measurable functions $g_k(x) \in L_\infty(0, \infty)$, $1 \leqslant k \leqslant r + 1$ such that

$$\langle f, \phi \rangle = \sum_{k=1}^{r+1} \langle g_k(x), \ P_{a,b}(x)\, x^{k-1} D^k \phi(x) \rangle.$$

This completes the proof.

Theorem 2 — Let $f \in H'_{a,b}$ and $\phi \in D(I)$, the space of all smooth functions with compact support on $I = (0, \infty)$. Then there exist M-bounded measurable functions $g_k(x) \in L_\infty(0, \infty)$, $1 \leqslant k \leqslant r + 1$ such that

$$\langle f, \phi \rangle = \sum_{k=1}^{r+1} \langle (-1)^k D^{k+1} \int_0^x P_{a,b}(t)\, g_k(t)\, dt, \ \phi \rangle. \qquad \ldots(4.7)$$

PROOF : Let $\phi \in D(I)$. Then in view of Theorem 1, there exist M-bounded measurable functions $g_k \in L_\infty(0, \infty)$, $1 \leqslant k \leqslant r + 1$ such that

$$\langle f, \phi \rangle = \sum_{k=1}^{r+1} \langle g_k, \ P_{a,b}(x)\, x^{k-1} D^k \phi(x) \rangle$$

$$= \sum_{k=1}^{r+1} \langle D \int_0^x P_{a,b}(t)\, t^{k-1}\, g_k(t)\, dt, \ D^k \phi \rangle$$

Since $\phi \in D(I)$, So also $D^k\phi$. Now we know that $P_{a,b}(x) x^{k-1}\, g_k(x)$ is locally integrable function on I.

$$\therefore \qquad D \int_0^x P_{a,b}(t)\, t^{k-1} g_k(t)\, dt = P_{a,b}(x)\, x^{k-1} g_k(x)$$

belongs to $D'(I)$ (Zemanian 1965, p. 54). Further this function is locally integrable on I. So it generates a regular distribution in $D'(I)$. i.e. $D \int_0^x P_{a,b}(t)\, t^{k-1} g_k(t)\, dt$ is a generalized function. Therefore on applying generalized differentiation (Zemanian 1965, p. 47) k times, we have

$$\langle f, \phi \rangle = \sum_{k=1}^{r+1} \langle (-1)^k D^{k+1} \int_0^x P_{a,b}(t)\, t^{k-1} g_k(t)\, dt, \ \phi(t) \rangle$$

or

$$f = \sum_{k=1}^{r+1} (-1)^k D^{k+1} \int_0^x P_{a,b}(t)\, t^{k-1} g_k(t)\, dt.$$

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