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PLANE SYMMETRIC SOLUTIONS OF INTERACTING SELF-GRAVITATING FLUIDS AND SOURCE-FREE ELECTROMAGNETIC FIELDS

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A class of exact solutions to Einstein's equations in general relativity for interacting self-gravitating irrotational fluids and source-free electromagnetic fields have been obtained for time-dependent plane-symmetric metric. The physical consequences of some of the solutions have been discussed. It has been observed that in the absence of source-free electromagnetic fields, one of our solutions reduces to the solution obtained by Sistero for nonstatic plane-symmetric zero-rest-mass scalar field equations.

1. INTRODUCTION

In recent years, self-gravitating fluid distributions have been studied in the frame-work of general theory of relativity by several authors' viz., Tabensky and Taub (1973) for plane symmetric metric, Letelier and Tabensky (1974) for Einstein-Rosen metric, Letelier (1975) and Ray (1976) for Marder's metric, Ray (1978) for conformally flat metric. In all these work exact solutions to Einstein's equations have been obtained under various physical assumptions. Wainwright *et al.* (1979) has considered the field equations correspond to the self-gravitating fluids $p = \rho$ in the frame-work of general relativity and have shown that the solution of these field equations admit a two parameter Abelian group of local isometries and represent inhomogeneous cosmological models. The present paper deals with finding out of exact solutions of the Einstein's field equations with interacting self-gravitating fluids and source-free electromagnetic fields as source. Though, the present work may be regarded as a special case of the work of Wainwright *et al.* (1979), yet the exact solutions found here may give some physical insight for the particular distribution.

It is well known, the results for self-gravitating fluid distributions in general relativity apply to the situation where the source of the gravitational field is a zero-rest-mass scalar field, since such a source has the same stress-energy tensor as an irrotational fluid with $p = \rho$. The problem therefore may also be looked upon as the problem of interacting zero-mass scalar fields and source-free electromagnetic fields, which have been studied separately by various authors (Rao *et al.* 1972, 1973). We have studied the problem for various cases, by considering different components of F_{ij} to exist. One of the solutions have been obtained by reducing the original metric with the help of characteristic coordinates, following Tabensky and Taub (1973).

It has been observed that one of our solutions, in the absence of electromagnetic fields, is identical with the solution obtained by Sistero (1976) for plane-symmetric zero-mass-scalar field distributions. Some of the solutions have been studied for their singular and other physical behaviour. All the solutions obtained, satisfy the reality conditions viz., the Hawking-Penrose condition (1970) and the energy condition.

2. Field Equations

The Einstein field equations for self-gravitating perfect fluid with pressure p equal to rest-energy density ρ and four velocity u_i are given by

$$R_{ij} = -\sigma_i \sigma_j \qquad \ldots(1)$$

$$\Box\, \sigma = (\sqrt{-g}\ \sigma_i g^{ij}), j = 0 \qquad \ldots(2)$$

when irrotationality is imposed viz.,

$$u_i = \frac{\sigma_i}{(\sigma_k \sigma^k)^{1/2}} \,. \qquad \ldots(3)$$

The pressure p and energy-momentum tensor T_{ij} are related to σ by

$$\rho = p = \tfrac{1}{2}\sigma_k \sigma^k$$

and

$$T_{ij} = 2\sigma_i \sigma_j - g_{ij}\sigma_k \sigma^k. \qquad \ldots(4)$$

We have chosen the units so that the velocity of light $c = 1$ and Newton's constant of gravitation $G = 1/8\pi$. A comma after an unknown function implies partial derivative w.r.t. the index. When the source contains, in addition to the self-gravitating fluid, source-free electromagnetic fields, the Einstein's equations become

$$R_{ij} = -\sigma_i \sigma_j - (g^{ab} F_{ai} F_{bj} - \tfrac{1}{4} g_{ij} F_{sp} F^{sp}) \qquad \ldots(5)$$

and

$$\Box\, \sigma = (\sqrt{-g}\ \sigma_i g^{ij})_{,j} = 0 \qquad \ldots(6)$$

with the Maxwell's equations

$$F_{ij} = A_{i,j} - A_{j,i}$$

and

$$F_{[ij;k]} = 0. \qquad \ldots(7)$$

These field equations for the plane-symmetric metric

$$ds^2 = e^{\alpha}(dt^2 - dx^2) - e^{\nu}(dy^2 + dz^2) \qquad \ldots(8)$$

where α and ν are functions of x and t only, assume the form

$$\alpha_{11} - \alpha_{44} + 2\nu_{11} + \nu_1^2 - \alpha_1\nu_1 - \alpha_4\nu_4 = -2\sigma_1^2 - e^{-\nu}(F_{12}^2 + F_{13}^2 + F_{24}^2 + F_{34}^2) - e^{-\alpha}F_{14}^2 \quad \ldots(9)$$

$$\nu_{44} - \nu_{11} + \nu_4^2 - \nu_1^2 = -e^{-\alpha}F_{14}^2 \quad \ldots(10)$$

$$\alpha_{44} - \alpha_{11} + 2\nu_{44} + \nu_4^2 - \alpha_4\nu_4 - \alpha_1\nu_1 = -2\sigma_4^2 - e^{-\nu}(F_{12}^2 + F_{13}^2 + F_{24}^2 + F_{34}^2) - e^{-\alpha}F_{14}^2 \quad \ldots(11)$$

$$2\nu_{14} + \nu_1\nu_4 - \alpha_1\nu_4 - \alpha_4\nu_1 = -2\sigma_1\sigma_4 + 2e^{-\nu}(F_{12}F_{24} + F_{13}F_{34}) \quad \ldots(12)$$

$$F_{12}F_{13} - F_{24}F_{34} = 0 \quad \ldots(13)$$

$$F_{12}^2 - F_{13}^2 = F_{24}^2 - F_{34}^2 \quad \ldots(14)$$

and

$$\sigma_{11} - \sigma_{44} + \sigma_1\nu_1 - \sigma_4\nu_4 = 0. \quad \ldots(15)$$

3. Solutions

We have solved the above equations for two different cases viz.,

(a) when $F_{14} = 0$ but F_{12}, F_{13}, F_{24} and F_{34} are nonzero;

(b) when $F_{14} \neq 0$ but other components are zero.

Case 3(*a*)

In this case we assume that $F_{14} = 0$ but F_{12}, F_{24}, F_{13} and F_{34} are nonzero. Let us denote $F_{12} = -\phi_1$, $F_{13} = -\psi_1$, $F_{24} = \phi_4$ and $F_{34} = \psi_4$. Then the reduced field equations are

$$(\alpha_{11} - \alpha_{44} + 2\nu_{11} + \nu_1^2 - \alpha_1\nu_1 - \alpha_4\nu_4) = -2\sigma_1^2 - e^{-\nu}(\phi_1^2 + \psi_1^2 + \phi_4^2 + \psi_4^2) \quad \ldots(16)$$

$$\nu_{44} - \nu_{11} + \nu_4^2 - \nu_1^2 = 0 \quad \ldots(17)$$

$$\alpha_{44} - \alpha_{11} + 2\nu_{44} + \nu_4^2 - \alpha_4\nu_4 - \alpha_1\nu_1 = -2\sigma_4^2 - e^{-\nu}(\phi_1^2 + \psi_1^2 + \phi_4^2 + \psi_4^2) \quad \ldots(18)$$

$$2\nu_{14} + \nu_1\nu_4 - \alpha_1\nu_4 - \alpha_4\nu_1 = -2\sigma_1\sigma_4 - 2e^{-\nu}(\phi_1\phi_4 + \psi_1\psi_4) \quad \ldots(19)$$

$$\phi_1\psi_1 - \phi_4\psi_4 = 0 \quad \ldots(20)$$

$$\phi_1^2 - \psi_1^2 = \phi_4^2 - \psi_4^2 \quad \ldots(21)$$

$$\sigma_{11} - \sigma_{44} + \sigma_1\nu_1 - \sigma_4\nu_4 = 0 \quad \ldots(22)$$

$$\phi_{11} - \phi_{44} = 0 \quad \ldots(23)$$

and

$$\psi_{11} - \psi_{44} = 0. \quad \ldots(24)$$

We have considered the following subcases to solve the above equations

3(a).1. $\phi = 0, \quad \psi \neq 0,$

3(a).2. $\phi \neq 0, \quad \psi = 0,$

3(a).3. $\phi \neq 0, \quad \psi \neq 0.$

Subcase 3(*a*).1

We have one of the solutions of eqn. (17) as given by

$$\nu = \log (k_1 x + k_2 t + k_3), \qquad \ldots(25)$$

where k_1, k_2, k_3 are arbitrary constants of integration. Using the above result, (16) and (18) yield the single equation

$$2(\alpha_{11} - \alpha_{44}) - \frac{k_1^2 - k_2^2}{(k_1 x + k_2 t + k_3)^2} = -2(\sigma_1^2 - \sigma_4^2). \qquad \ldots(26)$$

If we now consider

$$2\sigma_1^2 = \frac{k_1^2}{(k_1 x + k_2 t + k_3)^2}$$

and

$$2\sigma_4^2 = \frac{k_2^2}{(k_1 x + k_2 t + k_3)^2}$$

we get

$$\sigma = \frac{1}{\sqrt{2}} \log (k_1 x + k_2 t + k_3). \qquad \ldots(27)$$

The above value of σ satisfies the eqn. (22) identically. With this value of σ, (26) reduces to

$$\alpha_{11} - \alpha_{44} = 0. \qquad \ldots(28)$$

Substituting the value of σ and using (28), the eqns. (16) and (18) become

$$k_1 \alpha_1 + k_2 \alpha_4 = (\psi_1^2 + \psi_4^2). \qquad \ldots(29)$$

Also (21) and (24) yield

$$\psi = F(x \pm t).$$

The equations (29) and (19) then assume the form

$$k_1 \alpha_1 + k_2 \alpha_4 = 2F'^2 \qquad \ldots(30)$$

and

$$k_2 \alpha_1 + k_1 \alpha_4 = 2F'^2 \qquad \ldots(31)$$

respectively, where a prime denotes differentiation w.r.t. the argument $(x + t)$. From (30) and (31) we now have

$$(k_1 - k_2)(\alpha_1 - \alpha_4) = 0$$

which implies either $k_1 = k_2$ or $\alpha_1 = \alpha_4$.

Let us first consider $\alpha_1 = \alpha_4$, for which (30) and (31) reduce to

$$F'^2 = \left(\frac{k_1 + k_2}{2}\right)\alpha^1. \qquad \ldots(32)$$

This equation involves the two yet unsolved unknowns viz., α and ψ. Considering a solution of α (say), given by (28) the eqn. (32) can be solved for the other unknown ψ. Alternately, assuming a solution ψ given by (24), the eqn. (32) yields the unknown α. Thus for every value of ψ [satisfying (24)] or α [satisfying (28)] we can have different sets of solutions for ψ and α, the other unknowns (viz., ν and σ) are determined by (25) and (27).

Considering a well-known solution of (28) in the form

$$\alpha = \log(x + t) + a$$

we have, the final set of solution as given by

$$\alpha = \log(x + t) + a$$

$$\nu = \log(k_1 x + k_2 t + k_3)$$

$$\sigma = \frac{1}{\sqrt{2}} \log(k_1 x + k_2 t + k_3) \qquad \ldots(33)$$

and

$$\psi = \{2(k_1 + k_2)(x + t)\}^{1/2}.$$

Again, assuming two well-known solutions of (24) in the form

$$F = e^{a(x+t)}$$

and

$$F = \log(x + t) + a$$

we have two more sets of solutions as given by

$$\alpha = \frac{a}{k_1 + k_2} e^{2a(x+t)} + b,$$

$$\nu = \log(k_1 x + k_2 t + k_3)$$

$$\sigma = \frac{1}{\sqrt{2}} \log(k_1 x + k_2 t + k_3)$$

$$\psi = e^{a(x+t)} \qquad \ldots(34)$$

and

$$\alpha = b - \frac{2}{(k_1 + k_2)(x + t)},$$

$$\nu = \log (k_1 x + k_2 t + k_3),$$

$$\sigma = \frac{1}{\sqrt{2}} \log (k_1 x + k_2 t + k_3),$$

$$\psi = \log (x + t) + a, \qquad \ldots(35)$$

respectively. It is to be noted here and in what follows that the case $k_1 = k_2$ leads to the vanishing of the curvature invariant R.

Subcase 3(*a*).2

The case when $\phi \neq 0$ but $\psi = 0$, can be dealt with similarly.

Subcase 3(*a*).3

The solutions of (23) and (24) are respectively given by

$$\phi = f(x + t) + g(x - t)$$

and

$$\psi = F(x + t) + G(x - t).$$

Substituting the values of ϕ and ψ, eqns. (20) and (21) assume respectively the form

$$(f' + g')(F' + G') = (f' - g')(F' - G')$$

and

$$(f' + g')^2 - (F' + G')^2 = (f' - g')^2 - (F' - G')^2$$

which on simplification yield

$$\frac{f'}{F'} = -\frac{g'}{G'} \text{ and } \frac{f'}{F'} = \frac{G'}{g'}.$$

Thus, either

$$F'^2 + f'^2 = 0, \quad \text{or} \quad G'^2 + g'^2 = 0.$$

As the sum of two square terms equals to zero implies that the terms themselves are equal to zero, we have two possibilities viz.,

either F' and f' are zero or G' and g' are zero.

Thus we have the following two subcases

3(*a*).3.1. ϕ and ψ both are functions of the argument $(x + t)$;

3(*a*).3.2. ϕ and ψ both are functions of the argument $(x - t)$.

Subcase 3(*a*).3.1

For this case we assume $\phi = A\psi$. As in the subcase 3(*a*).1, we have from (17) and (22), the solutions of ν and σ as respectively given by

$$\nu = \log(k_1x + k_2t + k_3)$$

and

$$\sigma = \frac{1}{\sqrt{2}} \log(k_1x + k_2t + k_3).$$

With these values of ν and σ, arguing as in the subcase 3(*a*).1, we are finally left with the single equation

$$F'^2 = \frac{(k_1 + k_2)}{2(1 + A^2)} \alpha'.$$

Hence, for different values of α and F, we have the following sets of solutions:

$$\left.\begin{aligned} &\alpha = \log(x + t) + a, \ \nu = \log(k_1x + k_2t + k_3) \\ &\sigma = \frac{1}{\sqrt{2}} \log(k_1x + k_2t + k_3), \\ &\phi = A\left\{\frac{\sqrt{2}(k_1 + k_2)(x + t)}{(1 + A^2)}\right\}^{1/2}, \ \psi = \left\{\frac{\sqrt{2}(k_1+k_2)}{(1 + A^2)}(x + t)\right\}^{1/2} \end{aligned}\right\} \quad \ldots(36)$$

$$\left.\begin{aligned} &\alpha = \frac{a(1 + A^2)}{(k_1 + k_2)} e^{2a(x+t)} + b, \ \nu = \log(k_1x + k_2t + k_3) \\ &\sigma = \frac{1}{\sqrt{2}} \log(k_1x + k_2t + k_3), \ \psi = e^{a(x+t)}, \ \phi = A\psi \end{aligned}\right\} \quad \ldots(37)$$

and

$$\left.\begin{aligned} &\alpha = b - \frac{2(1 + A^2)}{(k_1 + k_2)(x + t)}, \ \nu = \log(k_1x + k_2t + k_3) \\ &\sigma = \frac{1}{\sqrt{2}} \log(k_1x + k_2t + k_3), \ \psi = \log(x + t) + a, \ \phi = A\psi \end{aligned}\right\}. \quad \ldots(38)$$

Case 3(*b*)

When only F_{14} is non-zero but other components of the electromagnetic field tensor are zero, the field equations are:

$$\alpha_{11} - \alpha_{44} + 2\nu_{11} + \nu_1^2 - \alpha_1\nu_1 - \alpha_4\nu_4 = -2\sigma_1^2 - e^{-\alpha}F_{14}^2 \quad \ldots(39)$$

$$\nu_{44} - \nu_{11} + \nu_4^2 - \nu_1^2 = -e^{-\alpha}F_{14}^2 \quad \ldots(40)$$

$$\alpha_{44} - \alpha_{11} + 2\nu_{44} + \nu_4^2 - \alpha_4\nu_4 - \alpha_1\nu_1 = -2\sigma_4^2 - e^{-\alpha}F_{14}^2 \quad \ldots(41)$$

$$2\nu_{14} + \nu_1\nu_4 - \alpha_1\nu_4 - \alpha_4\nu_1 = -2\sigma_1\sigma_4 \quad \ldots(42)$$

$$\sigma_{11} + \sigma_1\nu_1 - \sigma_{44} - \sigma_4\nu_4 = 0, \qquad \ldots(43)$$

$$\frac{\partial}{\partial x}\,(F^{14}\sqrt{-g}) = 0 \qquad \ldots(44)$$

and

$$\frac{\partial}{\partial t}\,(F^{41}\sqrt{-g}) = 0. \qquad \ldots(45)$$

Following Sistero (1976), assuming a functional relationship between the metric parameters in the form

$$\alpha = b - \nu \qquad \ldots(46)$$

and further assuming another functional relationship between σ and the metric parameter ν in the form

$$\sigma = c\nu + d \qquad \ldots(47)$$

we get after straight-forward calculations the final form of the solution as given by

$$\left.\begin{aligned} a\nu &= \log\{a(k_1x + k_2t) + k_3\}, \\ \alpha &= b - \nu,\ \sigma = c\nu + d, \\ F_{14} &= \left\{\frac{(1 + 2c^2)\,(k_2^2 - k_1^2)}{2}\right\}^{1/2} \frac{e^{\alpha/2}}{(a(k_1x + k_2t) + k_3)} \end{aligned}\right\} \qquad \ldots(48)$$

where

$$a = (2c^2 + 3)/2.$$

4. Some More Solutions

In this section, we present some more solutions obtained by transforming the original metric with the help of the characteristic coordinates given by

$$u = t - x \quad \text{and} \quad v = t + x. \qquad \ldots(49)$$

Using (49), the metric (8) reduces to the form

$$ds^2 = e^{\alpha}du\,dv - e^{\nu}(dy^2 + dz^2) \qquad \ldots(50)$$

where α and ν are now functions of u and v. The non-vanishing components of the Ricci-tensor for the metric (50) are

$$\begin{aligned} R_{uu} &= \nu_{uu} + \frac{\nu_u^2}{2} - \alpha_u\nu_u \\ R_{yy} \equiv R_{zz} &= -\frac{e^{\nu-\alpha}}{2}\,(\nu_{uv} + \nu_u\nu_v) \end{aligned}$$

(*equation continued on p.* 265)

$$\left.\begin{aligned} R_{vv} &= \nu_{vv} + \frac{\nu_v^2}{2} - \alpha_v\nu_v \\ \text{and} \\ R_{uv} &= \nu_{uv} + \tfrac{1}{2}\nu_u\nu_v + \alpha_{uv} \end{aligned}\right\} . \qquad ...(51)$$

In this case we obtain the solutions for the case when $F_{14} = 0$ but F_{12}, F_{13}, F_{24}, F_{34} are nonzero. Denoting $F_{12} = -\phi_u$, $F_{13} = -\psi_u$, $F_{24} = \phi_v$ and $F_{34} = \psi_v$, the Einstein field equations (5) – (7) become,

$$\nu_{uu} + \tfrac{1}{2}\nu_u^2 - \alpha_u\nu_u = -\sigma_u^2 + e^{-\nu}(\phi_u^2 + \psi_u^2) \qquad ...(52)$$

$$\nu_{vv} + \tfrac{1}{2}\nu_v^2 - \alpha_v\nu_v = -\sigma_v^2 + e^{-\nu}(\phi_v^2 + \psi_v^2) \qquad ...(53)$$

$$\nu_{uv} + \tfrac{1}{2}\nu_u\nu_v + \alpha_{uv} = -\sigma_u\sigma_v \qquad ...(54)$$

$$(e^{\nu})_{uv} = 0 \qquad ...(55)$$

$$\sigma_{uv} = -\tfrac{1}{2}(\sigma_u\nu_v + \sigma_v\nu_u) \qquad ...(56)$$

$$\frac{\partial}{\partial v}(\phi_u) + \frac{\partial}{\partial u}(\phi_v) = 0 \qquad ...(57)$$

$$\frac{\partial}{\partial v}(\psi_u) + \frac{\partial}{\partial u}(\psi_v) = 0 \qquad ...(58)$$

and

$$\phi_u\phi_v = \psi_u\psi_v. \qquad ...(59)$$

From (59), we have the following possible cases:

(4.1) ϕ is function of u and ψ is function of v

(4.2) ϕ is function of v and ψ is function of u

(4.3) ϕ and ψ both are functions of u and v.

Case (4.1)

Equation (55) has a general solution given by

$$\nu = f(u) + g(v).$$

For convenience, we consider the case when $\nu = f(u) + A$ where f is an arbitrary function of u only. The eqn. (56) on integration yields

$$\sigma = Be^{-\nu/2}v + \eta(u) \qquad ...(60)$$

where $\eta(u)$ is any arbitrary function of u and B is an integration constant.

From (54) we obtain α as given by

$$\alpha = c - Bv \int e^{-\nu/2}\eta_u du - \tfrac{1}{4}B^2v^2e^{-\nu}. \qquad ...(61)$$

Now using the above values of σ, ν and α in (52) and (53), ϕ and ψ are easily obtained and the solution finally assumes the form

$$\left.\begin{aligned}
&\alpha = c - \tfrac{1}{4}B^2v^2e^{-\nu} - Bv \int e^{-\nu/2}\eta_u du, \\
&\nu = A + f(u),\ \sigma = Bve^{-\nu/2} + \eta(u),\ \psi = Bv + D, \\
&\phi = \int e^{\nu/2}.(\nu_{uu} + \tfrac{1}{2}\nu_u^2 + \eta_u^2)^{1/2}\, du,
\end{aligned}\right\} \quad \ldots(62)$$

where A, B, C and D are constants of the integration.

Case (4.2)

The case when ϕ is a function of v and ψ is a function of u can be dealt with similarly.

Case (4.3)

From eqns. (57) and (58) we have

$$\phi = f(u) + g(v) \quad \text{and} \quad \psi = F(u) + G(v).$$

Again from (59), we get

$$f_u g_v + F_u G_v$$

which implies

$$\frac{f_u}{F_v} = \frac{G_u}{g_v} = A \text{ (constant).}$$

Thus we have

$$\phi = f(u) + g(v) \text{ and } \psi = \frac{1}{A} f(u) + Ag(v).$$

From (55), we get

$$\nu = \eta(u) + \mu(v).$$

Considering as before ν to be a function of u only i.e.,

$$\nu = \eta(u) + B \quad \ldots(63)$$

we get from (56), after a straight-forward calculation

$$\sigma = cve^{-\nu/2} + h(u).$$

Using the value of ν and σ in (54), we have

$$\alpha = D - \tfrac{1}{4}c^2v^2e^{-\nu} - cv \int h_u e^{-\nu/2}\, du.$$

The functions f and g determining ϕ and ψ are then obtained from (49) and (50) with the help of ν, α, σ in the form

$$f(u) = \frac{A}{(1 + A^2)^{1/2}} \int (\nu_{uu} + \tfrac{1}{2}\nu_u^2 + h_u^2)^{1/2} e^{\nu/2} du \qquad \ldots(64)$$

and

$$g(v) = \frac{1}{(1 + A^2)^{1/2}} cv + E.$$

Hence, the final form of the solution is given by

$$\alpha = D - \tfrac{1}{4}c^2v^2e^{-\nu} - cv \int h_u e^{-\nu/2} du,$$

$$\nu = f(u) + B, \quad \sigma = cv + e^{-\nu/2} + h(u),$$

$$\phi = \frac{A}{(1 + A^2)^{1/2}} \int (\nu_{uu} + \tfrac{1}{2}\nu_u^2 + h_u^2)^{1/2} e^{\nu/2} du + \frac{cv}{(1 + A^2)^{1/2}} + E,$$

$$\psi = \frac{1}{(1 + A^2)^{1/2}} \int (\nu_{uu} + \tfrac{1}{2}\nu_u^2 + h_u^2)^{1/2} e^{\nu/2} du + \frac{Ac}{(1 + A^2)^{1/2}} v + F. \qquad \ldots(65)$$

5. Discussions

It may be observed for all the solutions presented in section 3, that σ and ν remain the same and depending on the choice of ϕ and (or) ψ, the other metric parameter α is affected. Alternately choosing α properly, the electromagnetic components ϕ and ψ are determined.

The pressure p given by

$$p = \tfrac{1}{2}\, \sigma_s \sigma^s$$

for the metric (8) reduces to the form

$$p = \tfrac{1}{2}\, e^{-\alpha}(\sigma_4^2 - \sigma_1^2).$$

For, the solutions of section 3 it can be easily observed that at the source, the pressure in having a finite value and it tends to $\rightarrow 0$ as $x \rightarrow \infty$. The four velocity vector u^4 given by $u^4 = \sigma^4/(\sigma_s\sigma^s)^{1/2}$ is also having a similar limiting behaviour as the pressure.

The solutions represent divergent gravitational and self-gravitating fields σ as either $x \rightarrow \infty$ or $t \rightarrow \infty$ or both.

The solutions satisfy the relation $\omega \equiv [(F_{ij}F^{ij})^2 + (F_{ij}F^{*ij})^2] = 0$, implying thereby that they represent null electromagnetic fields. It may be observed that in the absence of the electromagnetic field one of the solutions reduces to that of the solution for plane-symmetric zero-mass scalar fields obtained by Sistero (1976).

We have observed that the two reality conditions viz., the Hawking-Penrose condition $(T_{ab} - \tfrac{1}{2}\, Tg_{ab})\, u^a u^b \geqslant 0$ and the energy condition $T_{ab}u^a u^b \geqslant 0$ are satisfied for our solutions.

For the physical distribution considered, the eigenvalues, characterized by the determinantal equations

$$| T^i_j - \lambda g^i_j | = 0,$$

are found and it has been observed that two of the eigenvalues are equal and the other two are equal and opposite. It may be noted here, that the presence of the electromagnetic field does not alter the eigenvalue behaviour of the solutions.

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References

Hawking, S. W., and Penrose, R. (1970). The singularities of gravitational collapse and cosmology. *Proc. R. Soc.*, **A314**, 529.

Letelier, P. S. (1975). Self-gravitating fluids with cylindrical symmetry. *J. Math. Phys.*, **16**, 1488.

Letelier, P. S., and Tabensky, R. (1974). Cylindrically self-gravitating fluids with pressure equal to energy density. *Il Nuovo Cim.*, **28** B(2), 407.

Ray, D. (1976). Self-gravitating fluids with cylindrical symmetry. *J. Math. Phys.*, **17**, 1171.

———— (1978). Self-gravitating irrotational barotropic fluid in a conformally flat space. *J. Math. Phys.*, **19**, 3.

Rao, J. R., Roy, A. R., and Tiwari, R. N. (1972–73). A class of exact solutions for coupled electromagnetic and scalar fields for Einstein-Rosen metric I. *Ann. Phys.* (*N. Y.*), **69**, 473; **78**, 553; **79**, 276.

Sistero, R. F. (1976). A nonstatic plane-symmetric solution of Einstein's field equaiton for zero-rest-mass scalar fields. *G. R. G.*, **7**, 6, 487.

Tabensky, R., and Taub, A. H. (1973). Plane-symmetric self-gravitating fluids with pressure equal to energy density. *Comm. Math. Phys.*, **29**, 61.

Wainwright, J., Ince, W. C. W., and Marshman, B. J. (1979). Specially homogeneous and inhomogeneous Cosmologies with equation of state $p = \mu$. *G.R.G.*, **10**, 259.