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THE STRUCTURE OF ROTATING POLYTROPES

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The structure of rapidly rotating polytropic models, with polytropic indices $n = 1.5, 2$ and 3 have been discussed by using Chandrasekhar's (1933), perturbation techniques for the inner solution and Monaghan and Roxburgh's approximation method (1965) for the outer solution. The solution have been fitted at the points $\xi_f = 3.025, 3.470$ and 4.950 for $n = 1.5$, 2 and 3 respectively. These interfacial points have been chosen to minimise the difference between values of neglected terms of inner and outer solution at the interfaces. This has ensured more accurate results than what Monaghan and Roxburgh's have obtained.

1. INTRODUCTION

A considerable work has been done on the structure of uniformly rotating polytropes (Chandrasekhar 1933, Monaghan and Roxburgh 1965, James 1964, Naylor and Anand 1970). Chandrasekhar (1933) developed a first order perturbation method to solve the basic equations of rotation. This method actually does not give accurate results near the surface of a model. Monaghan and Roxburgh (1965, hereafter it will be referred as MR method) used Chandrashekhar's first order expansion technique in the inner region and an approximation technique in the outer region by neglecting the mass of the outer layers. The two solutions were then matched at a suitable interface. The best choice of the interfacial point is determined by the condition that the errors of inner and outer solutions should be of the same order at the fitting point. In order to fulfil this condition we have recalculated the polytropic models with indices $n = 1.5$, 2 and 3. This has given rise to new fitting points (Monaghan and Roxburgh 1965) for all the three models.

2. BASIC EQUATIONS AND METHOD OF SOLUTION

The basic equations governing the structure of polytropes of index n, rotating with angular velocity Ω, can be expressed as

$$\left.\begin{aligned} \frac{\nabla P}{\rho} &= -\nabla\Phi + \Omega^2\bar{w} \\ P &= K\rho^{(1+n)/n} \\ \nabla^2\Phi &= 4\pi\rho G \end{aligned}\right\} \qquad \ldots(1)$$

where the various symbols have the same meaning as given in MR (1965).

In terms of dimensionless variables ξ and α

$$\rho = \rho_c \sigma^n; \; r = a\xi; \; \alpha = \frac{\Omega^2}{2\pi G \rho_c}$$

$$a = \left[\frac{(n+1)\, K \rho^{(1-n)/n}}{4\pi G}\right]^{1/2}$$

the basic equation (1) transforms to

$$\nabla^2_{\xi,\mu}\, \sigma = \frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial \sigma}{\partial \xi}\right) + \frac{1}{\xi^2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial \sigma}{\partial \mu}\right] = -\sigma^n + \alpha \qquad \ldots(2)$$

with boundary conditions

$$\sigma = 1, \; \frac{\partial \sigma}{\partial \xi} = \frac{1}{\xi}\frac{\partial \sigma}{\partial \mu} = 0 \quad \text{at } \xi = 0.$$

In the inner region the perturbation forces are small compared to local gravitational force and here we use a first order perturbation technique developed by Chandrasekhar (1933) for the solution.

σ is expanded in power series as

$$\sigma = \theta + \alpha\Psi + \alpha^2\chi + \ldots .$$

Substituting in eqn. (2) and equating the coefficients of α, we get

$$\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial \Psi}{\partial \xi}\right) + \frac{1}{\xi^2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial \Psi}{\partial \mu}\right] = -n\theta^{n-1}\Psi + 1. \qquad \ldots(3)$$

Let $\Psi = \sum_{m=0}^{\infty} A_m \psi_m P_m(\mu)$. Substituting in (3) and equating coefficients of P_m in (3) we obtain

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\psi_0}{d\xi}\right) = -n\theta^{n-1}\psi_0 + 1, \quad A_0 = 1$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\psi_m}{d\xi}\right) - \frac{m(m+1)}{\xi^2}\psi_m = -n\theta^{n-1}\psi_m, \; m \neq 0$$

with boundary conditions

$$\psi_m = \frac{d\psi_m}{d\xi} = 0 \quad \text{at} \quad \xi = 0.$$

The inner solution is now in terms of the parameter A_m.

The perturbation method is valid only for the inner regions as long as $|\alpha\psi| < \theta$. A different approximation technique is needed to solve the eqn. (2) in the outer region as $|\alpha\psi|$ will not be less than θ near the surface. We will use MR (1965) method to solve eqn. (2) near the surface.

Let the gravitational potential Φ be defined as

$$\Phi = -K(n+1)\,\rho_c^{1/n}\,\phi.$$

Eliminating P from the basic eqn. (1) the resulting equations after the integration gives

$$\sigma = \phi + \tfrac{1}{6}\alpha\xi^2(1 - P_2(\mu)) + \nu \ldots \qquad \ldots(4)$$

where ν is a constant of integration.

The other relations between ϕ and σ is given by

$$\nabla^2_{\xi,\mu}\,\phi = -\sigma^n. \qquad \ldots(5)$$

If we neglect the density near the surface then ϕ is given by

$$\sigma_0^2 = 0;\ \nabla^2_{\xi,\mu}\,\phi_0 = -\sigma_0^n = 0. \qquad \ldots(6)$$

The solution of eqn. (6) helps to get a better first order approximation from the eqns. (4) and (5) as

$$\sigma_1 = \phi_0 + \tfrac{1}{6}\alpha\xi^2\,(1 - P_2(\mu)) + \nu \ldots . \qquad \ldots(7)$$

Under the condition that ϕ tends to zero as ξ tends to infinity, the eqn. (7) gives

$$\sigma_1 = \frac{a_0}{\xi} + \frac{a\xi^2}{6}[1 - P_2(\mu)] + \nu + \sum_{m=1}^{\infty}\frac{\alpha a_m P_m(\mu)}{\xi^{(m+1)}}. \qquad \ldots(8)$$

The outer solution for σ in (8) is known in terms of the parameter a_m.

3. At the Interface

The interface at $\xi = \xi_f$ is chosen in such a way that ξ_f gives maximum accuracy and this is achieved by the condition that error of both the regions are equal at the interface. The maximum accuracy is obtained by ensuring that the neglected terms are of comparable magnitude i.e.

$$\left[\frac{\alpha\Psi}{\theta}\right]^2 = 1 - \left[\frac{M(\xi_f)}{M}\right]. \qquad \ldots(9)$$

A maximum value of α corresponding to equatorial break up is chosen to get the suitable value of ξ_f, which is approximately given by

$$\alpha = 0.36\,(\bar{\rho}/\rho_c). \qquad \ldots(10)$$

Ψ is taken as ψ_0 calculated by Chandrasekhar's (1933) method and the values of $M(\xi_f)/M$ are taken from non-rotating polytropes.

With the starting value of α obtained from (10), the fitting constants λ_0, λ_1, ν_0, ν_1, a_2, A_2 (see Appendix) were obtained. Then these were substituted in eqns. (A1) and (A2) of appendix to obtain a new improved values of α and the equatorial radius ξ_e. Then for this new value of α, $\left[\frac{\alpha\psi}{\theta}\right]^2$ was recalculated and new fitting radius ξ_f was chosen to reduce the difference between the values of $\left[\frac{\alpha\psi}{\theta}\right]^2$ and $\beta = 1 - \frac{M(\xi_f)}{M}$. With the new fitting radius, the constants λ_0, λ_1, ν_0, ν_1, a_2, A_2; were recalculated and by substituting in the eqns. (A1) and (A2) of the appendix, a new value of α and ξ_e were calculated. This new value of α was used to calculate $\left[\frac{\alpha\psi}{\theta}\right]^2$ and a new fitting radius ξ_f was chosen to ensure that ξ_f is a root of eqn. (9).

In Table I(a) we have given the functional values θ, θ', ψ_0, ψ_0', ψ_2, ψ_2'. In the Table I(b) we have given the values of $\left[\frac{\alpha\psi}{\theta}\right]^2$ and β at the fitting radius ξ_f. In Table II, are given the values of various constants. Table III gives the values of α, ξ_e and ξ_p. These values have been compared with MR (1965), James (1964) and Naylor and Anand (1970). Table IV gives the values of ξ_e and ξ_p for various values of α. In Table V are given the values of σ_0, σ_1 and σ_2 for n = 1.5, 2 and 3.

TABLE I(a)

n	θ	θ'	ψ_0	ψ_0'	ψ_2	ψ_2'	ξ_f
1.5	0.151791	−0.281091	0.957832	0.464403	4.12860	1.081081	3.025
2.0	0.139761	−0.194440	1.2072942	0.631467	4.373569	1.2220o2	3.47
3.0	0.114864	−0.081680	2.657948	1.231347	6.410483	1.962316	4.95

TABLE I(b)

n	$\beta = 1 - [M(\xi_f)/M]$	$\left[\frac{\alpha\psi}{\theta}\right]^2$	ξ_f	(ξ_f) $MR^* NA^*$
1.5	0.052284	0.052373	3.025	3.2
2.0	0.027977	0.027977	3.47	3.6
3.0	0.008357	0.008367	4.95	5.0

*Monaghan and Roxburgh (1965).

**Naylor and Anand (1970).

TABLE II

Values of constants at interfaces

n	λ_0	λ_1	ν_0	ν_1	A_2	a_2	ξ_f
1.5	2.572159	4.977305	−0.698509	−2.212663	−.4870645	−13.446934	3.025
2.0	2.3412506	6.3238758	−0.5349511	−2.621965	−.5787457	−27.00069	3.47
3.0	2.0013911	10.258047	−0.28945694	−3.4981348	−.70589992	−53.17589992	4.95

TABLE III

n	1.5	2.0	3.0
α(NA)	0.037543	0.019439	0.0039304
α^*	0.036179	0.019363	0.0039527
α(J)	0.043	0.021604	0.003932
α(MR)	0.041	0.0199	0.00395
ξ_e(NA)	5.3687	6.3748	10.1370
ξ_e^*	5.35380	6.35038	10.11451
ξ_e(J)	5.3585	6.307	—
ξ_e(MR)	5.24	6.33	10.12
ξ_p^*	3.48388	4.15941	6.62394
ξ_p(J)	3.30	4.06	6.58
ξ_p(MR)	3.49	4.15	6.72

*Present
NA : Naylor and Anand (1970)
J : James (1964)
MR : Monaghan and Roxburgh (1965).

TABLE IV

$n = 1.5$

α	ξ_p^*	ξ_e^*	$\xi_p(J)^{**}$	$\xi_e(J)^{**}$
		$n = 1.5$		
0	3.68235	3.68235	3.6538	3.6538
0.02	3.5685527	4.0577289	3.4968	3.9896
0.024	3.5469889	4.1739264	3.4645	4.0818
0.028	3.5257911	4.3220358	3.4317	4.1808
0.032	3.5049465	4.532138	3.3983	4.3174
0.036	3.4844426	4.957447	3.3642	4.4815

(*Table* IV *continued on p.* 270)

TABLE IV (*continued from p.* 269)

α	ξ^*_p	ξ^*_e	$\xi_p(J)^{**}$	$\xi_e(J)^{**}$
		$n = 2.0$		
0	4.376569	4.396569	4.35288	4.35288
0.002	4.3522199	4.4411076	4.32595	4.41383
0.006	4.3045661	4.5910009	4.27203	4.55256
0.010	4.2582387	4.782652	4.21763	4.72315
0.014	4.2131613	5.0535111	4.16267	4.94728
0.018	4.1692627	5.5580029	4.10680	5.28708

α	ξ^*_p	ξ^*_e	$\xi_p(J)^{**}$	$\xi_e(J)^{**}$	$\xi_p(mR)^{***}$	$\xi_e(mR)^{***}$
			$n = 3.0$			
0	6.9142974	6.9142974	6.89685	6.89685	6.87	7.20
0.001	6.8628967	7.1850455	6.84227	7.15980	6.87	7.20
0.002	6.8125961	7.549064	6.78772	7.50803	6.82	7.56
0.003	6.7633571	8.1154606	6.73301	8.03338	6.77	8.14
0.0038	6.7247054	9.1372499	6.68918	8.87126	6.73	9.19
0.0039	6.7199191	9.4760084	—	—	6.73	9.57

*Present
**James (1964)
***Monaghan and Roxburgh (1965).

TABLE V

ξ	σ_0	σ_1	σ_2
	$n = 1.5$		
0	1	0.000	0.000
0.4	0.973651	0.02635048	−0.076591
0.8	0.898277	0.10178539	−0.291250
1.2	0.783977	0.21668937	−0.603678
1.6	0.644850	0.35861624	−0.961209
2.0	0.495937	0.51611558	−1.1313986
2.4	0.350487	0.68195448	−1.627018
2.8	0.218192	0.85515470	−1.886366
3.2	0.105290	1.049412	−2.117037
3.6	0.015979	1.329922	−2.448216
	$n = 2.0$		
0	1	0.0	0.0
0.4	0.973754	0.0252926	−0.090501
0.8	0.899797	0.09901046	−0.338898
1.2	0.790670	0.20966203	−0.687668
1.6	0.662363	0.34438722	−1.071448
2.0	0.529836	0.49462778	−1.440896
2.4	0.404210	0.65830987	−1.772959
2.8	0.291988	0.83948332	−2.068480
3.2	0.195727	1.0467703	−2.344101
3.6	0.115396	1.294667	−2.626494
4.0	0.050362	1.625670	−3.006740

(*Table V continued on p.* 271)

TABLE V (*continued from p.* 270)

ξ	σ_0	σ_1	σ_2
		$n = 3.0$	
0	1	0.0	0.0
0.4	0.973958	0.02604951	−0.109173
0.8	0.902672	0.0977696	−0.397227
1.2	0.802592	0.20151946	−0.777625
1.6	0.691544	0.32663002	−1.174236
2.0	0.582851	0.47022729	−1.550013
2.4	0.483928	0.63593044	−1.901476
2.8	0.397589	0.83063766	−2.242018
3.2	0.323914	1.0616636	−2.589365
3.6	0.261694	1.3362204	−2.95967
4.0	0.209282	1.6604482	−3.365792
4.4	0.165033	2.0395175	−3.817386
4.8	0.127281	2.4776726	−4.321496
5.2	0.095426	2.981233	−4.884854
5.6	0.067934	3.560325	−5.529464
6.0	0.044108	4.211538	−6.246184
6.4	0.023260	4.931349	−7.029515
6.8	0.004865	5.717065	−7.875781

CONCLUSIONS

We have obtained ξ_f in eqn. (9) as near the exact root values ξ of this equation [Table I(a)]. This has ensured more accurate structure values. The results of the model $n = 1.5$, in particular have shown remarkable accuracy in ξ_e as is evident from Table III.

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APPENDIX

On the fitting surface $\xi = \xi_f$, we impose the conditions that σ and $\partial\sigma/\partial\xi$ be continuous, implying that $\partial\sigma/\partial\mu$, ϕ, $\partial\phi/\partial\xi$ and $\partial\phi/\partial\mu$ are continuous. This gives

$$\theta + \alpha\psi_0 = \frac{a_0}{\xi} + \frac{\alpha\xi^2}{6} + \nu$$

$$\alpha A_2\psi_2 = \frac{\alpha a_2}{\xi^3} - \frac{a\xi^2}{6}$$

$$\alpha A_m\psi_m = \frac{\alpha a_m}{\xi^{m+1}}, m \neq 0, 2$$

$$\frac{d\theta}{d\xi} + \frac{\alpha d\psi_0}{d\xi} = -\frac{a_0}{\xi^2} + \frac{\alpha\xi}{3}$$

$$\alpha A_2 \frac{d\psi_2}{d\xi} = \frac{-3\alpha a_2}{\xi^4} - \frac{\alpha\xi}{3}$$

$$\alpha A_m \cdot \frac{d\psi_m}{d\xi} = -\frac{(m+1)\, a_m\alpha}{\xi^{(m+2)}}, \quad m \neq 0, 2$$

at $\xi = \xi_f$. To solve the equations we take $a_0 = \lambda_0 + \alpha\lambda_1$,

$$\nu = \nu_0 + \alpha\nu_1.$$

The final solution is now of eqn. (3) as

$$\sigma = \sigma_0(\xi) + \alpha\sigma_1(\xi) + \alpha\sigma_2(\xi)\,(P_2(\mu))$$

where

$$\xi \leqslant \xi_f;\ \sigma_0 = \theta,\ \sigma_1 = \psi_0,\ \sigma_2 = A_2\psi_2$$

$$\xi \geqslant \xi_f;\ \sigma_0 = \frac{\lambda_0}{\xi} + \nu_0,\ \sigma_1 = \frac{\lambda_1}{\xi} + \frac{\xi^2}{6} + \nu_1$$

$$\sigma_2 = \frac{a_2}{\xi^3} - \frac{\xi^2}{6}.$$

The Critical Configuration

For a particular value of α the polytrope will be on the verge of equatorial break-up, and the centrifugal force will balance gravity at the equator.

This will occur when

$$\sigma = 0,\ \frac{\partial\sigma}{\partial\xi} = 0 \text{ at } \mu = 0.$$

This gives

$$\frac{\lambda_0}{\xi} + \frac{\alpha\lambda_1}{\xi} - \frac{\alpha a_2}{2\xi^3} + \frac{\alpha}{4}\xi^2 + \nu_0 + \alpha\nu_1 = 0 \qquad \text{...(A1)}$$

$$-\frac{\lambda_0}{\xi^2} - \frac{\alpha\lambda_1}{\xi^2} + \frac{3\alpha a_2}{2\xi^4} + \frac{\alpha\xi}{2} = 0. \qquad \text{...(A2)}$$