Indian J. pure appl Math., 13(3) : 273–286, March 1982

SOME MATHEMATICAL MODELS FOR OPTIMAL MANAGEMENT OF FORESTS

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(*Received* 5 *June* 1981)

Some linear and non-linear mathematical models for determining the optimum sowing and harvesting management policies for forests have been formulated and solved with the help of techniques of mathematical programming.

1. INTRODUCTION

There exists a vast and growing body of literature on the optimal harvesting of age-structured animal populations. Doubleday (1975) gave a linear programming model for determining the maximum sustainable yield from such a population. Rorres (1976) generalized the model to include k linear constraints where $k < n$, the number of age-groups. He showed that the optimal policy is such that there can be at most k age-groups which are partially harvested and other age-groups have to be completely harvested or not harvested at all. Earlier Beddington and Taylor (1973) and Rorres and Fair (1975) had arrived at the same conclusion for the case $k = 1$, without using the linear programming formulation.

Kapur (1979c, d, e) considered the possibility when population sizes are not just sustained, but grow so that the population of each group, after growth in a time period and being harvested at the end of it, is λ times its value at the beginning of the time period ($1 < \lambda < \lambda_0$) where λ_0 is the positive dominant eigenvalue of the growth matrix of the model of Lewis (1942) and Leslie (1945). Kapur (1980b) also considered the optimization problems for the density-dependent non-linear models age-structured population models of Leslie (1959) and Kapur (1979a, 1980a). For these non-linear models, he required non-linear programming techniques, but in each case, by a suitable substitution, he could reduce the non-linear programming problem to a set of linear programming problems. Kapur (1979b) also developed a continuous-time discrete-age-scale model for age-structured populations, in terms of systems of differential equations and formulated and solved the above optimization problems for this model.

Rorres and Anton (1977) gave an elementary model for optimal management of forests on the same lines as those of Doubleday (1975). Instead of age-groups of animals, height-groups of trees were considered and instead of new animals being born, planting of new seedlings to offset the effect of harvesting of trees was considered.

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In the present paper, we generalize their model for

(a) obtaining steadily growing yield from forests, taking the discount factor into account;

(b) non-linear density-dependent models;

(c) continuous time discrete-age-scale models;

(d) effects of limited space available for a forest.

2. Model for Steady Growth of Forest Wealth

Let $x_i(t)$ be the number of trees in the ith height-group $(i = 1, 2, \ldots, n)$ at time t. Let g_i be the proportion of trees in the ith age-group which grow to become trees of $(i+1)$th height-group in one period so that a proportion $(1 - g_i)$ of the trees continues to remain in the ith height-group. Let $y_i(t)$ be the number of trees removed from the ith group at the end of this period. Let P_i be the profit on a tree of the ith group. Also let

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}, \quad P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix}, \quad I_* = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \ldots(1)$$

$$G = \begin{bmatrix} 1-g_1 & 0 & 0 & \ldots & 0 & 0 \\ g_1 & 1-g_2 & 0 & \ldots & 0 & 0 \\ 0 & g_2 & 1-g_3 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & 1-g_{n-1} & 0 \\ 0 & 0 & 0 & \ldots & g_{n-1} & 1 \end{bmatrix},$$

$$R = \begin{bmatrix} 1 & 1 & 1 & \ldots\ldots & 1 & 1 \\ 0 & 0 & 0 & \ldots\ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots\ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots\ldots & 0 & 0 \end{bmatrix} \qquad \ldots(2)$$

so that $I'_* X = x_1 + x_2 + \ldots + x_n; \quad I'_* Y = y_1 + y_2 + \ldots + y_n$ ...(3)

Here $X(t)$ is the population vector, $Y(t)$ the harvesting vector, P the profit vector, G the growth matrix and R the replacement matrix. For increasing forest wealth, we plant μ times the trees we harvest. The population vector X becomes GX due to growth, is reduced by Y by harvesting, is increased by μRY by planting and should finally be λX so that we get

$$GX - Y + \mu RY = \lambda X \qquad \text{...(4)}$$

or $$(G - \lambda I) X = (I - \mu R) Y. \qquad \text{...(5)}$$

Multiplying (4) by I'_*, we get

$$I'_* GX - I'_* Y + I'_* \mu RY = \lambda I'_* X$$

or $$I'_* X - I'_* Y + \mu I'_* Y = \lambda I'_* X$$

or $$(\mu - 1) I'_* Y = (\lambda - 1) I'_* X \qquad \text{...(6)}$$

which means that the number of additional seedlings planted give the number of additional trees in the next period.

From (4) and (6)

$$GX - Y + \left[1 + (\lambda - 1) \frac{I'_* X}{I'_* Y}\right] RY = \lambda X. \qquad \text{...(7)}$$

We take $y_1 = 0$ since it is no use planting new seedlings and then removing them, then (8) gives

$$\left.\begin{aligned} (1 - g_1) x_1 - y_1 + \left[1 + (\lambda - 1) \frac{x_1 + x_2 + \dots + x_n}{y_1 + y_2 + \dots + y_n}\right] & \\ \times [y_1 + y_2 + \dots + y_n] = \lambda x_1 & \\ g_1 x_1 + (1 - g_2) x_2 - y_2 = \lambda x_2 & \\ g_2 x_2 + (1 - g_3) x_3 - y_3 = \lambda x_3 & \\ g_{n-1} x_{n-1} + (1 - g_n) x_n - y_n = \lambda x_n & \end{aligned}\right\} \qquad \text{...(8)}$$

where $y_1 = 0$, $g_n = 0$. Adding these, we get an identity, so that only the last $(n-1)$ equations are independent. These give the profit function

$$P_2 [g_1 x_1 + (-\lambda + 1 - g_2) x_2] + P_3 [g_2 x_2 + (-\lambda + 1 - g_3) x_3] + \dots + P_n [g_{n-1} x_{n-1} + (-\lambda + 1) x_n]. \qquad \text{...(9)}$$

This has to be maximized subject to

$$g_i x_1 + (-\lambda + 1 - g_{i+1}) x_{i+1} \geqslant 0; \quad i = 1, 2, \dots, n - 1 \qquad \text{...(10)}$$

$$x_j \geqslant 0; \quad j = 1, 2, \dots, n \qquad \text{...(11)}$$

and $$x_1 + x_2 + \dots + x_n = s, \qquad \text{...(12)}$$

where s is the size of the forest at the beginning of the first period.

This size becomes λs, $\lambda^2 s$, $\lambda^3 s$, ... at the beginnings of second, third, fourth,...periods respectively.

This is the standard linear programming problem. According to the theorem of Rorres (1976), with at most one exception, either every population is completely harvested or is not harvested at all. Moreover if the kth height-group is harvested, the higher height-group will not exist

If only the kth group is harvested

$$x_k = x_{k+1} = \dots = x_n = 0\,; \quad x_1 + x_2 + \dots + x_{k-1} = s \qquad \dots(13)$$

and

$$\left.\begin{aligned} g_1 x_1 &= (\lambda - 1 + g_2)\, x_2 \\ g_2 x_2 &= (\lambda - 1 + g_3)\, x_3 \\ &\cdots\cdots \\ g_{k-2} x_{k-2} &= (\lambda - 1 + g_k)\, x_{k-1}. \end{aligned}\right\} \qquad \dots(14)$$

These give the vector X where

$$X = \begin{bmatrix} \frac{1}{g_1}\left(1 + \frac{\lambda - 1}{g_2}\right)\left(1 + \frac{\lambda - 1}{g_3}\right) \dots \left(1 + \frac{\lambda - 1}{g_{k-2}}\right)(g_{k-1} + \lambda - 1) \\ \frac{1}{g_2}\left(1 + \frac{\lambda - 1}{g_3}\right) \dots \left(1 + \frac{\lambda - 1}{g_{k-2}}\right)(g_{k-1} + \lambda - 1) \\ \cdots\cdots\cdots \\ \frac{1}{g_{k-3}}\left(1 + \frac{\lambda - 1}{g_{k-2}}\right)(g_{k-1} + \lambda - 1) \\ \frac{1}{g_{k-2}}(g_{k-1} + \lambda - 1) \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} x_{k-1}, \qquad \dots(15)$$

where

$$\left[\frac{(g_2 + \lambda - 1) \dots (g_{k-1} + \lambda - 1)}{g_1 g_2 \cdots g_{k-2}} + \frac{(g_3 + \lambda - 1) \dots (g_{k-1} + \lambda - 1)}{g_2 g_3 \cdots g_{k-2}} + \dots + \frac{g_{k-1} + \lambda - 1}{g_{k-2}} + 1\right] x_{k-1} = s. \qquad \dots(16)$$

The profit function is given by

$$\begin{aligned} P_k y_k &= P_k g_{k-1} x_{k-1} \\ &= P_k g_{k-1} s \end{aligned}$$

(equation continued on p. 277)

$$= \frac{P_k S}{\left[\frac{1}{g_{k-1}} + \frac{1}{g_{k-2}}\left(1+\frac{\lambda-1}{g_{k-1}}\right) + \frac{1}{g_{k-3}}\left(1+\frac{\lambda-1}{g_{k-1}}\right)\left(1+\frac{\lambda-1}{g_{k-2}}\right) + \ldots + \frac{1}{g_1}\left(1+\frac{\lambda-1}{g_{k-1}}\right)\ldots\left(1+\frac{\lambda-1}{g_2}\right)\right]} \qquad \ldots(17)$$

$= f(k, \lambda)$ (say).

If $\lambda > 1$, the lower-height groups get greater weightage and the profit is reduced for every k.

The present value of profits over a time span of m periods is obtained by multiplying the expression in (17) by

$$\frac{1-(\mu\lambda)^m}{1-\mu\lambda} \qquad \ldots(18)$$

where μ is the discount factor for a period. We first choose k to maximize (17) for a given value of λ and then choose λ so as to maximize the product of this maximum and (18) i.e. maximum profit is obtained as

$$\max_{\lambda} \left\{\frac{1-(\mu\lambda)^m}{1-\mu\lambda}\,[\max_{k}\; f(k, \lambda)]\right\}. \qquad \ldots(19)$$

3. Non-linear Models

If the number of trees in the forest goes on increasing, the density-dependent effects can become important and each g_i has now to be considered as a decreasing function of the total population size

$$N = x_1 + x_2 + \ldots + x_n. \qquad \ldots(20)$$

We have still to maximize (9) subject to (10) and (11), but now g_i's are functions of N so that we have apparently a non-linear objective function and non-linear constraints. However for a fixed N, the problem is still of linear programming

For a given N, the profit function is

$$g(k, N, \lambda) = \frac{P_k N}{\left[\frac{1}{g_{k-1}(N)} + \frac{1}{g_{k-2}(N)}\left(1+\frac{\lambda-1}{g_{k-1}(N)}\right) + \ldots + \frac{1}{g_1(N)}\left(1+\frac{\lambda-1}{g_{k-1}(N)}\right)\ldots\left(1+\frac{\lambda-1}{g_2(N)}\right)\right]} \qquad \ldots(21)$$

For each value of N, we find the value of k for which this is maximum and then find the maximum profit as a function of N. Then we choose N in such a way that this maximum profit is maximized. Alternatively, since both k and N may be treated as integers, we may use finite differencing to choose k_0 and N_0 in such a way that

$$\left.\begin{aligned} g(k_0 - 1, N_0, \lambda) < g(k_0, N_0, \lambda);\ g(k_0 + 1, N_0, \lambda) < g(k_0, N_0, \lambda) \\ g(k_0, N_0 - 1, \lambda) < g(k_0, N_0, \lambda);\ g(k_0, N_0 + 1, \lambda) < g(k_0, N_0, \lambda). \end{aligned}\right\} \quad ...(22)$$

If we also allow λ to vary, the maximum profit is given by

$$\max_{\lambda} \left\{ \max_{N} [\max_{k} g\,(k, N, \lambda)] \right\}. \quad ...(23)$$

After the forest has reached a saturation size we may take $\lambda = 1$, to get

$$h(k, N) = g(k, N, 1) = \frac{P_k N}{\left[\dfrac{1}{g_{k-1}(N)} + \dfrac{1}{g_{k-2}(N)} + \ldots + \dfrac{1}{g_1(N)}\right]}. \quad ...(24)$$

The simplest case arises if, on the analogy of the density-dependent model of Leslie (1959), we take

$$g_i(N) = \frac{g_i}{1 + aN} \quad (i = 1, 2, \ldots, k - 1) \quad ...(25)$$

so that

$$h(k, N) = \frac{N}{1 + aN} \frac{P_k}{\left[\dfrac{1}{g_{k-1}} + \dfrac{1}{g_{k-2}} + \ldots + \dfrac{1}{g_1}\right]} \quad ...(26)$$

and the same group is harvested as in the absence of density-dependence. If however we take the density-dependent model of Kapur (1980a), to consider

$$g_i(N) = \frac{g_i}{1 + (a_0 + a_1 + \ldots + a_i)N}; \quad i = 1, 2, \ldots, k - 1 \quad ...(27)$$

the group harvested depends on both k and N.

4. Continuous-time Models

The forest-harvesting and forest-plantation need not be once-a-year operations, but can be carried on almost continuously in which case we can use the continuous-time discrete-age-scale model

$$\left.\begin{aligned} \frac{dx_1}{dt} &= y_1 + y_2 + y_3 + \ldots + y_n - g_1 x_1 \\ \frac{dx_2}{dt} &= g_1 x_1 - g_2 x_2 - y_2 \\ \frac{dx_3}{dt} &= g_2 x_2 - g_3 x_3 - y_3 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{dx_{n-1}}{dt} &= g_{n-2} x_{n-2} - g_{n-1} x_{n-1} - y_{n-1} \\ \frac{dx_n}{dt} &= g_{n-1} x_{n-1} - y_n \end{aligned}\right\} \quad ...(28)$$

so that since $y_1 = 0$

$$\frac{dx_1}{dt} + \frac{dx_2}{dt} + \dots + \frac{dx_n}{dt} = 0 \quad \dots(29)$$

and the number of trees in the forest remains constant.

In the steady case $dx_i/dt = 0$ for all i and we have to maximize

$$P_2y_2 + P_3y_3 + \dots + P_ny_n = P_2(g_1x_1 - g_2x_2) + \dots + P_ng_{n-1}x_{n-1} \quad \dots(30)$$

subject to

$$\left.\begin{array}{l} g_1x_1 \geqslant g_2x_2 \geqslant g_3x_3 \geqslant \dots \geqslant g_{n-1}x_{n-1} \\ x_1 \geqslant 0,\ x_2 \geqslant 0 \quad \dots;\ x_n \geqslant 0 \end{array}\right\}. \quad \dots(31)$$

This is the linear programming problem solved by Rorres and Anton (1977). If we want the forest population to grow steadily, we write

$$\frac{dx_i}{dt} = (\lambda - 1)\, x_i \quad ; \qquad \lambda > 1 \quad \dots(32)$$

and set $\quad x_i = x_{i0}\, e^{(\lambda-1)t} \quad , \qquad y_i = y_{i0}\, e^{(\lambda-1)t} \quad \dots(33)$

to get

$$\left.\begin{array}{l} (y_{20} + y_{30} + \dots + y_{n0}) = (\lambda - 1 + g_1)\, x_{10} \\ y_{20} = g_1x_{10} + (1 - \lambda - g_2)\, x_{20} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_{n-1,0} = g_{n-2}x_{n-2,0} + (1 - \lambda\, g_{n-1})\, x_{n-1,0} \\ y_{n,0} = g_{n-1}x_{n-1,0} + (1 - \lambda)\, x_{n,0}. \end{array}\right\} \quad \dots(34)$$

If δ is the instantaneous discount rate, the present value of profits is

$$\int_0^\infty e^{-\delta t}(P_2y_2 + P_3y_3 + \dots + P_ny_n)\, dt$$

$$= \int_0^\infty e^{(\lambda-1-\delta)t}\, (P_2y_{20} + P_3y_{30} + \dots + P_ny_{n0})\, dt$$

$$= \frac{1}{\delta + 1 - \lambda}\, [P_2y_{20} + P_3y_{30} + \dots + P_ny_{n0}] \quad \dots(35)$$

provided $\delta + 1 - \lambda > 0$ or $1 < \lambda < 1 + \delta$. Even if $\lambda > 1 + \delta$, but we have to consider a finite span of time, we have to maximize

$$P_2\,[g_1x_{10} + (1 - \lambda - g_2)\, x_{20}] + P_3\,[g_2x_{20} + (1 - \lambda - g_3)\, x_{30}]$$
$$+ \dots + P_n\,[g_{n-1}x_{n-1,0} + (1 - \lambda)\, x_{n,0}] \quad \dots(36)$$

subject to coefficients of P_i's and x_i's being all $\geqslant 0$

This is the same optimization problem as was considered in section 2.

If in (28), g's are functions of $x_1 + x_2 + \dots + x_n$ and we consider the steady case ($\lambda = 1$), we get the same optimization problems as for the discrete-time model.

5. Constraint on Available Forest Space

If A is the total available forest space and $A_1, A_2, \dots, A_n$ are the areas required for trees of different height groups, then we get the constraint equation

$$A_1 x_1 + A_2 x_2 + \dots + A_n x_n = A. \qquad \dots(37)$$

In this case according to the theorem of Rorres (1976), the kth species will be completely harvested and the $(k-1)$th will be partially harvested. We consider the case $\lambda = 1$ and let h be the fraction of the $(k-1)$th species harvested, then we get

$$\left.\begin{aligned} &g_1 x_1 - g_2 x_2 = 0 \\ &g_2 x_2 - g_3 x_3 = 0 \\ &g_{k-3} x_{k-3} - g_{k-2} x_{k-2} = 0 \\ &g_{k-2} x_{k-2} - g_{k-1} x_{k-1} = h x_{k-1} = y_{k-1} \\ &g_{k-1} x_{k-1} = y_k \\ &x_k = x_{k+1} = \dots = x_n = 0. \end{aligned}\right\} \qquad \dots(38)$$

Using (13) and (38) we get

$$\frac{x_1}{\frac{1}{g_1}} = \frac{x_2}{\frac{1}{g_2}} = \dots = \frac{x_{k-2}}{\frac{1}{g_{k-2}}} = \frac{x_{k-1}}{\frac{1}{g_{k-1} + h}} = \frac{s}{\frac{1}{g_1} + \frac{1}{g_2} + \dots + \frac{1}{g_{k-1} + h}} \qquad \dots(39)$$

and

$$\frac{A_1}{g_1} + \frac{A_2}{g_2} + \dots + \frac{A_{k-2}}{g_{k-2}} + \frac{A_{k-1}}{g_{k-1} + h} = \frac{A}{s}\left(\frac{1}{g_1} + \frac{1}{g_2} + \dots + \frac{1}{g_{k-1}+h}\right)$$

or

$$\frac{1}{g_1}(\bar{A} - A_1) + \frac{1}{g_2}(\bar{A} - A_2) + \dots + \frac{1}{g_{k-1} + h}(\bar{A} - A_{k-1}) = 0, \qquad \dots(40)$$

where $\bar{A}$ is the average space available per tree. Equation (40) determines h

Profit function $= P_{k-1} y_{k-1} + P_k y_k$

$$= P_{k-1} h x_{k-1} + P_k g_{k-1} x_{k-1}$$

$$= (P_{k-1} h + P_k g_{k-1}) \frac{s \times \frac{1}{g_{k-1} + h}}{\frac{1}{g_1} + \frac{1}{g_2} + \dots + \frac{1}{g_{k-1} + h}}$$

(*equation continued on p.* 281)

$$= [(P_k - P_{k-1})\, g_{k-1} + P_{k-1}\,(g_{k-1} + h)]$$

$$\times \left[\frac{s}{1 + (g_{k-1} + h)\left[\frac{1}{g_1} + \frac{1}{g_2} + \ldots + \frac{1}{g_{k-2}}\right]} \right]. \qquad \ldots(41)$$

Substituting from (40) from $g_{k-1} + h$ in (41) we get the profit function as dependent upon k and we can find the value of k for which the profit function is maximum.

6. Generalisation of His Earlier Models by Rorres

In a recent paper, Rorres (1978)* has further generalised his earlier models, (see Rorres and Fair 1975, Rorres 1976, Rorres and Anton 1977), which were themselves generalisations of the pioneering models for forest management of Usher (1966; 1969a, b; 1976) in the following directions:

(a) Instead of allowing a tree to grow to the next size-class (height-class or girth-class) only, he allows for the possibility of its growing by two or more size-classes in one harvesting period. Thus a tree in the ith class at the end of a harvesting period may have belonged to the first, second,... or ith class at the beginning of this period. This means that the matrix G in (2) is replaced by matrix $[g_{ij}]$ where g_{ij} is now the probability of a tree growing from size-class j to size class i in one harvesting period. Here $g_{ij} \geqslant 0$ and $g_{ij} = 0$ when $j > i$, so that $[g_{ij}]$ is a lower triangular matrix.

(b) He further breaks up matrix $[g_{ij}]$ into two matrixes $L = [a_{ij}]$ and $D = [b_{ij}]$ where a_{ij} and b_{ij} are respectively the probabilities of a tree being live or dead at the end of the period when it grows from jth class to ith class in that period. This distinction is made because all dead trees must be harvested and the economic values of dead and live trees can be different.

(c) He replaces the scalar μ by the vector $(r_1, r_2, \ldots, r_n)$ so that the recruitment matrix R is replaced by $\bar{R}$ where the first row-vector of $\bar{R}$ is $(r_1, r_2, \ldots, r_n)$ and all other row-vectors are zero vectors. This means that when a tree of ith class is harvested, it is replaced by $r_i (\geqslant 0)$ seedlings.

The vector X of live trees at the beginning of a harvesting period becomes LX of live trees and DX of dead trees at the end of the period. Vectors Y of live trees and DX of dead trees are removed and replaced by $R(\bar{Y} + DX)$ new plants so that in the case of sustained yield, we get

$$LX - Y + \bar{R}(Y + DX) = X. \qquad \ldots(42)$$

The economic value of the yield is

$$V = PY + QDX \qquad \ldots(43)$$

*The author is sincerely grateful to the referee for bringing this paper to his attention.

where P and Q are the profit vectors for live and dead trees respectively.
Using (42), this becomes

$$V = P[I - \bar{R}]^{-1}[L + \bar{R}D - I]X + QDX. \qquad ...(44)$$

Rorres (1978) considered the linear programming problem of maximizing V subject to $X \geqslant 0$ and K normalising and other constraints of the form

$$v_{i1}x_1 + v_{i2}x_2 + \ldots + v_{in}x_n = s_i;\ i = 1, 2, \ldots, K. \qquad ...(45)$$

The earlier model of Rorres (see Rorres and Fair 1975, Rorres 1976, Rorres and Anton 1977) can be obtained by putting

$$D = 0,\ \mu = I_*,\ L = G. \qquad ...(46)$$

In this case also, the optimal harversting policy has the property that 'at most' K classes of live trees are partially harvested. In each of the remaining classes, the live trees are either completely harvested or are not harvested at all. If a tree can advance by one size-class only in one period, only one size-class has to be harvested completely and in this case instead of using the revised simplex algorithms of linear programming, an alternative algorithm explained in the next section can be used.

7. Possible Generalisations of Our Earlier Models

In sections 2–5, we have generalised the model of Rorres and Fair (1975), Rorres (1976), and Rorres and Anton (1977) in the following directions:

(a) While they consider optimal sustainable yields, we have considered optimal yields which increase at a sustained rate.

(b) While they confine themselves to linear models only, we have considered non-linear density-dependent models also.

(c) While they confine themselves to discrete-time discrete-age-scale models, we have also considered continuous-time discrete-age-scale models.

(d) We have also considered the effects of discount rates which become important for management over long periods of time.

(e) We have also considered the effects of limited forest space in the context of sustained increasing growth.

Rorres (1978) did not consider any one of these. If we consider each of these along with his three generalisations, we get fifteen possibilities. Of course all these are not of equal importance. We have also to remember that each generalisation may add to the complexity of the model.

8. Optimal Sustained-rate Yields

For this case (42) and (44) are replaced by

$$LX - Y + \bar{R}(Y + DX) = \lambda X, \qquad \lambda \geqslant 1 \qquad ...(47)$$

and $$V = P[I - \bar{R}]^{-1}[L + \bar{R}D - \lambda I]X + QDX. \quad \ldots(48)$$

For the case, when trees advance by at most one size-class only, only one size-class has to be completely harvested. Let it be the kth class, then we seek solutions of the form

$$X = (x_1, x_2, \ldots, x_{k-1}, 0, 0, \ldots 0),\ Y = (y_1, y_2, \ldots, y_{k-1}, y_k, 0, 0, \ldots, 0),$$
$$y_k \neq 0 \quad \ldots(49)$$

so that from (47)

$$a_{11}x_1 - y_1 + (r_1y_1 + r_2y_2 + \ldots + r_ny_n) + r_1b_{11}x_1$$
$$+ r_2(b_{21}x_1 + b_{22}x_2) + \ldots + r_{k-1}(b_{k-1,k-2}x_{k-2} + b_{k-1,k-1}x_{k-1})$$
$$+ r_kb_{k,k-1}x_{k-1} = \lambda x_1 \quad \ldots(50)$$

$$\left.\begin{array}{l} a_{21}x_1 + a_{22}x_2 - y_2 = \lambda x_2 \\ a_{31}x_1 + a_{32}x_2 - y_3 = \lambda x_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{k-1,k-2}x_{k-2} + a_{k-1,k-1}x_{k-1} - y_{k-1} = \lambda x_{k-1} \end{array}\right\} \quad \ldots(51)$$

$$a_{k,k-1}x_{k-1} - y_k = 0\,. \quad \ldots(52)$$

We have also the constraints

$$v_{i1}x_1 + v_{i2}x_2 + \ldots + v_{i,k-1}x_{k-1} = s_i,\ i = 1, 2, \ldots, K. \quad \ldots(53)$$

Equations (50)–(53) give $k + K$ equations to determine.

$$x_1, x_2, \ldots, x_{k-1}; \qquad y_k; \qquad y_1y_2, \ldots, y_{k-1} \quad \ldots(54)$$

In general, we can determine K of $y_1, y_2, \ldots, y_{k-1}$. We can thus keep K of these as non-zero. This is consistent with the result of Rorres (1976) that at most K of the size-groups can be partially harvested. Thus if there is only one constraint, only one of $y_1, y_2 \ldots y_{k-1}$ can be non-zero, the others are zero. If y_i is the non-zero element, the profit is

$$V_{i,k} = P_iy_i + P_ky_k + Q_1b_{11}x_1 + Q_2(b_{21}x_1 + b_{22}x_2) + \ldots$$
$$+ Q_{k-1}(b_{k-1,k-1}x_{k-2} + b_{k-1,k-1}x_{k-1}) + Q_kb_{k,k-2}x_{k-1}$$
$$(i = 1, 2, \ldots, k - 1) \quad \ldots(55)$$

We can first fix k and find $V_{i,k}$ for $i = 1, 2, \ldots, k - 1$ and then we can vary k and find the values of i, k which give the optimal economic value. This algorithm avoids the use of the revised simplex algorithms.

For the numerical example of Rorres which he solved by using revised simplex algorithms on an IBM 5100 computer, we have

$$\left.\begin{array}{l} a_{11} = 0.67,\ a_{21} = 0.23,\ a_{22} = 0.64,\ a_{32} = 0.26,\ a_{33} = 0.70,\ a_{43} = 0.20, \\ a_{44} = 0.22,\ a_{54} = 0.18,\ a_{55} = 0.58,\ a_{65} = 0.32,\ a_{66} = 0.58 \end{array}\right\} \quad \ldots(56)$$

$$b_{12} = b_{21} = b_{22} = b_{32} = b_{33} = b_{43} = b_{44} = b_{54} = b_{55} = b_{64}$$
$$= b_{65} = 0.05,\ b_{66} = 0.15 \quad \ldots(57)$$

$$r_1 = r_2 = r_3 = 0, \quad r_4 = 3.60, \quad r_5 = 5.10, \quad r_6 = 7.10 \quad \ldots(58)$$

$$\left.\begin{array}{l} v_1 = P_1 = Q_1 = 1,\ v_2 = P_2 = Q_2 = 1,\ v_3 = P_3 = Q_3 = 4, \\ v_4 = P_4 = Q_4 = 9,\ v_5 = P_5 = Q_5 = 16,\ v_6 = P_6 = Q_6 = 25, \\ S_1 = 100{,}000,\ K = 1,\ \lambda = 1. \end{array}\right\} \quad \ldots(59)$$

The maximum economic value is obtained when $k = 4, i = 1$ so that eqns. (50–54) give

$$\left.\begin{array}{l} .67x_1 = y_1 + 3.60y_4 + 3.60(.05)\,x_3 = x_1 \\ .23x_1 + .64x_2 = x_2 \\ .26x_2 + .70x_3 = x_3 \\ .20x_3 = x_4 \end{array}\right\} \quad \ldots(60)$$

which give the solution

$$\left.\begin{array}{l} x_1 = 25949,\ x_2 = 16579,\ x_3 = 14368;\ y_1 = 4368,\ y_4 = 2874 \\ V_{1,4} = 46309 \end{array}\right\} \quad \ldots(61)$$

obtained by Rorres (1978).

In special cases, no size-class may be partially harvested. The condition for this may be obtained by putting $y_1 = y_2 = \ldots = y_{k-1} = 0$ in (50) and (51) and eliminating $x_1, x_2, \ldots, x_{k-1}, y_k$ from the $(k + 1)$ eqns. (50), (51), (52) and one of (53).

9. Other Generalisations

(a) *Non-linear Models*

In this case a_{ij} and b_{ij} are functions of the total population size $N = x_1 + x_2 + \ldots + x_n$ so that (48) is a non-linear function and we have a problem of non-linear programming. However for a fixed size N, it is still a linear programming problem. We solve this problem for each value of N and then find the value of N which maximizes the economic yield.

It can be shown by using the discrete maximum principle that even for the non-linear case, the optimal policy requires that with at most K exceptions, every size-class is either completely harvested or is not harvested at all.

(b) *Continuous-time Discrete-age-scale Model*

In this case, eqns. (28) are modified to

$$\frac{dx_1}{dt} = r_1y_1 + r_2y_2 + \ldots + r_ny_n - (a_{21} + a_{31} + \ldots + a_{n1})\, x_1$$

$$- (b_{21} + b_{31} + \ldots + b_{n1})\, x_1 - y_1$$

$$\frac{dx_2}{dt} = (a_{21} + b_{21})\, x_1 - (a_{32} + a_{42} + \ldots + a_{n2})\, x_2 - (b_{32} + b_{42} +$$

$$\ldots + b_{n2})\, x_2 - y_2$$

$$\frac{dx_3}{dt} = (a_{31} + b_{31})\, x_2 + (a_{32} + b_{32})\, x_2 - (a_{43} + a_{53} + \ldots + a_{n3})\, x_3$$

$$- (b_{43} + b_{53} + \ldots + b_{n3})\, x_3 - y_3$$

....

$$\frac{dx_n}{dt} = (a_{n1} + b_{n1})\, x_1 + (a_{n2} + b_{n2})\, x_2 + \ldots$$

$$+ (a_{n,n-1} + b_{n,n-1})\, x_{n-1} - y_n. \qquad \ldots(62)$$

In the steady-case $dx_i/dt = 0$ for all i, we can solve for $y_1, y_2, \ldots, y_n$ from (62) in terms of $x_1, x_2, \ldots, x_n$ and maximize $P_1y_1 + P_2y_2 + \ldots + P_ny_n$ subject to $x_i \geqslant 0$, $y_i \geqslant 0$ and whatever other constraints may be imposed in the system. We get linear programming problems when a's, b's are constants and non-linear programming problems when these are functions of $x_1 + x_2 + \ldots + x_n$.

(c) *Non-steady Case*

For the non-steady case, we have to maximize

$$P.\ V. = \int_0^{\infty} e^{-\delta t}\,[P_1y_1 + P_2y_2 + \ldots + P_ny_n]\, dt \qquad \ldots(63)$$

where $y_1, y_2, \ldots, y_n$ are obtained from (62) in terms of $x_1, x_2, \ldots, x_n$ and $x_1', x_2', \ldots, x_n'$ subject to the constraints that may be given. Now

$$P.\ V. = \int_0^{\infty} e^{-\delta t}\ [\sum_{i=1}^{n} A_ix_i + \sum_{i=1}^{n} B_ix_i']\, dt \qquad \ldots(64)$$

where A_i's, B_i's are constants or functions of $x_1 + x_2 + \ldots + x_n$ according as the model is linear or density-dependent.

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